Structure in Real Banach Spaces

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A list of corrections for "Structure in real Banach spaces" by Erik M. Alfsen and Edward G. Effros.

Page	line	as in text	replacement
iv	10	Theorem 7.6	Theorem 7.8
	11b	Theorem 7.7	Theorem 7.9 (see Remark 7.10).
	2b	pre-dual V	pre-dual U .
1.1	12	$p \leq q$ if	$p \leq q$ mod C (or simply $p \leq q$) if
	11b	proper cone	proper convex cone
	6b	the extreme	the set of extreme
1.2	5b	Cor. I.15	Cor I.1.15
	1b	w*-topology	weak*-topology
1.3	12b	$P_o(K)$	$P_p(K)$
	2b	$\gamma < f(P)$	$\gamma \leq f(P)$
1.5	8	[A_2, p.22]	[A_2, p.23]
	3b	[Ch 1]	[Ch]
	3b	[A_2, §2]	[A_2, Ch I, §2]
1.6	3	convex	linear
2.2	2	$p \mapsto \|q\|$	$p \mapsto \|p\|$
	6	proper cone	facial cone
	6b	let f be	let q be
2.4	5b	$0 < p_i < q_i$	$0 \leq p_i \leq q_i$
2.7	8	$\gamma < \delta, \delta'$	$\gamma \leq \delta, \delta'$
	7b	$p \prec p_\gamma$	$p_\gamma \succ p$
2.8	1b	$p_n \|p_n\|$	$p_n / \|p_n\|$
2.9	13	convex space	convex Hausdorff space
2.10	1	the unit	the closed unit
	1	A(K)	A(H)
	7	$0 \leq \alpha < 1$	$0 \leq \alpha \leq 1$
	6b	Finally --- $= \emptyset$	If $\alpha = 0$, then $r \in D_K^c \cap H = D_H^c$, and we are done.
	2b	by Ellis also	by Asimow and Ellis
	1b	[E_1, p.3]	[As-El, p.302]
3.2	11	L' summand	L-summand
3.3	12b	es er	es $\prec$ er
3.5	3	s	S
3.7	2	(3.6)	(3.5)
3.8	12b	w*-compact	weak* compact

page	line	as in text	replacement
3.8	7b	of ,	of $\mathcal{D}$,
3.11	9	$(T+\epsilon)e \leq Se$	$(T+\epsilon I)e \leq Se$
	9	$(T+\epsilon)e \prec Te$	$(T+\epsilon I)e \prec Te$
	11	$\|T+\epsilon)p\|$	$\|(T+\epsilon I)p\|$
3.12	4	w_γ	e_γ
3.13	11b	onto	onto. Since $\|S_1\| = \|S\|$, it is isometric.
	9b	$C(N)$	$C(W)$
	6b	f_1W	f_1N
3.14	8	$\vartheta(N)$	$\vartheta(N_o)$
	7b	N	$N \neq \{0\}$
	2b	and	and if $N_i \neq \{0\}$, $i = 1,2$, then
3.15	3-4	It follows $\cdots \subset S$	Assuming that $N_i \neq \{0\}$, $E(N \cap K) \subseteq S$, and if $p \in E(N \cap K)$,
3.16	1	n_2	N_2
4.1	10	all linear	all continuous linear
	10	i.e. affine	i.e. continuous affine
	5b	[La$_1$]	[La]
4.3	12	$= \alpha_1 v(p_1)$	$\geq \alpha_1 v(p_1)$
4.4	4-7	Noting that ..., a contradiction	We have that $1 = \|p\| \leq \|q_1\|+\|q_2\| \leq \alpha_1+\alpha_2+\alpha_3 = 1$, hence $q_1 \prec p$. Since N' is hereditary, $q_1 \in N' \cap N = \{0\}$, a contradiction.
	10	$(1-e)W$	$(I-e)W$
4.8	1-3	(replace 3 first lines with	$\|p\| \leq \|ep\|+\|p-ep\| \leq \|\mu\|_{K\cap N}\|+\|\mu\|_{K\cap N}\| \leq \|\mu\| = \|p\|$.
	7	W	N
	2b	v	ν
5.5	3	<u>Proof</u>. Let	<u>Proof</u>. We may select $\epsilon_1 < \epsilon$ with $b \leq g+\epsilon_1$. Let
	4	$g+\epsilon$	$g+\epsilon_1$
	7	$(g \wedge b)^\vee$	$(\check{g} \wedge b)^\vee$
5.6	1	$b(p_2 < g(p_2)+\epsilon$:	$b(p_2) \leq g(p_2)+\epsilon_1$, and the linearity of b ,
	2	$\eta >$	$\eta \geq$
	2	$\geq -\delta\epsilon$.	$\geq -\delta\epsilon_1 > -\delta\epsilon$.
5.7	10b	$a \leq 1$	$\|a\| \leq 1$
	6b	spaces it	spaces, i.e. Banach spaces with L^1 dual, it

page	line	as in text	replacement
5.7	4b	[La$_1$]	[La]
5.11	8	$< g$	$\leq g$
5.16	1b	[P-C]	[C-P]
5.17	8	(see § 7)	(see § 9)
	3b	(5.12) and (5.13)	(5.30) and (5.31)
	2b	(5.11)	(5.29)
5.18	3	Theorem 5.3	Theorem 5.4
5.21	1	5.15	5.14
	5	$\Phi_{J_1 \cap J_2} \cdots \Phi_{J_2}(v)$.	$\lVert v+J_1 \cap J_2 \rVert = \max\{\lVert v+J_1 \rVert, \lVert v+J_2 \rVert\} = \lVert v+J_2 \rVert$.
5.22	14	5.16	5.15
5.22a	7	$(I - e^*)v$	$(I - e)v$
	4b	5.17	5.16
	1b	5.16	5.15
6.1	3b	(3.12)	Proposition 3.14 (a)
6.2	6	Section 7	Section 9
6.3	2b	§ 8	Section 9
6.6	14	Prim $V \cdots \neq \emptyset$.	Prim $V \setminus h(J) \cong$ Prim $J \neq \{0\}$.
6.7	3	Proposition 5.19	Corollary 5.20
6.7	7	weak	weak*
	8	$E(K) \setminus N_p$	$E(K) \cap N_p$
	14	Z	$Z \setminus \{0\}$
	1b	weak	weak*
7.4	10	(7.2)	Corollary 7.4
	5b	7.3	7.5
7.5	8b	7.4	7.6
7.6	5	If we	If $p \neq 0$ and we
7.7	4b	7.5	7.7
7.8	8	_Tv)	(Tv)
	2b	7.6	7.8
7.9	11b	7.4	7.6
7.11	2	7.6	7.8
	3	Theorem.	Theorem for C^* algebras (see Section 10).
	4	7.7	7.9
	7	7.6	7.8
	11	$\tilde{T} \geq 1$	$\tilde{T}(p) \geq 1$
	10b	7.6	7.8
	1b	$U_o V_o$	$U_o V_o p$

page	line	as in text	replacement
7.14	5-6	i.e. that... v(Sp)	and it suffices to prove that for each $v \in V$ the function $v \circ S$
	8	$v(S_n p)$	$v \circ S_n$
	9	$v(S_n p)$	$v \circ S_n$
	4b	E(K).	E(K) (see the discussion following (7.13))
8.1	11	[A,...,W]	[As_1],[C-P],[El_3],[N_2],[Pe_2],[W_1].
	11b	D_2	D_1
8.3	2b	$-p^p$	$-p^-$
8.5	15	$a \geq 0$	$\alpha \geq 0$
	12b	[Buck]	[Buc]
8.7	7b	$p^+ \in N$	$p \vee 0 \in N$
	6b	$r \in N^+$	$q,r \in N^+$
	6-5b	$p^{\pm} \leq q$ (for ... ... and p). Thus	$p \vee 0 \leq q$. Thus
	2b	D_2	Day
9.4	3	9.2	9.1
	11	7.4	7.6
9.6	10	(C)	——
	9b	$u,s \in V^+$	$u,v \in V^+$
	1b	5.16	5.15
9.7	11b	$u,v \quad D^+$	$u,v \in D^+$
9.8	5b	[Pe_2, p.80]).	[Pe_2, p.80] or [G, p.556]).
	3b	[Pe_2,...,p.556]	[D-S, p.429]
9.11	8	5.16	5.15
	8b	5.16	5.15
9.12	1	$\{0\}$	$\setminus\{0\}$
10.2	6b	[La_1]	[La]
	1b	[H]	[A-H]
10.3	11b	[W]	[W_1]
	8b	the latter	the former
	2b	summands,	summands (see also [Pe_2]),
10.4	13	M-ideal	M-summand
	13	N^o	N
	13	in W^*	in W

page	line	as in text	replacement
10.5	13	Add these to the bibliography:	B_4: N. Bourbaki, Espaces vectoriels topologiques. Ch.III-V, Act.Sci. et Ind. 1229, Paris 1955. D-S: N. Dunford and J. Schwartz, Linear operators, Interscience, New York, 1958 Sa_2: S. Sakai, A characterization of W*-algebras, Pac.J. Math 6 (1956), 763-773.
ii	10	Ch_1	Ch
iv	7	La_1	La
	16	L_1	L
v	3	Sa	Sa_1

INTRODUCTION

In this paper we study a real Banach space by means of geometric and analytic properties of the closed unit ball of the dual space. The central theme of the paper is the investigation of certain subspaces, called M-ideals, which are analogous to (and in fact generalize) the closed two-sided ideals in a C*-algebra.

Suppose that V is a real Banach space, and that W is the dual Banach space of V. We say that a subspace J of V is an M-summand of V if there is a subspace H of V with $J \cap H = \{0\}$, $J + H = V$ and for each $j \in J$, $h \in H$,

$$\|j+h\| = \max\{\|j\|, \|h\|\} .$$

Similarly we say that a subspace N of W is an L-summand of W if there is a subspace M with $N \cap M = \{0\}$, $N + M = W$, and for each $p \in N$, $q \in M$,

$$\|p+q\| = \|p\| + \|q\| .$$

Finally, a closed subspace J of V is said to be an M-ideal of its annihilator J^o is an L-summand in W. These subspaces include but are far more extensive than the M-summands.

We say that a subspace J of V satisfies the n-ball property if given n open balls $B_1,\ldots,B_n$ for which $B_1 \cap \ldots \cap B_n \neq \emptyset$, and $B_i \cap J \neq \emptyset$, $i = 1,\ldots,n$, it follows that $B_1 \cap \ldots \cap B_n \cap J \neq \emptyset$.

The first five sections are, in part, devoted to a proof of the following result:

<u>Theorem A</u>. Suppose that J is a closed subspace of V. Then the following are equivalent:

(a) J is an M-ideal.

(b) J satisfies the 3-ball property.

Examining figure (i), it is evident that no one-dimensional subspace J of the Euclidean plane $\mathbb{R}^2$ is an M-ideal, since even two balls can intersect but fail to have mutual intersection with J. On the other hand, if one provides $\mathbb{R}^2$ with the norm

$$\|(\alpha_1,\alpha_2)\| = \max\ \{|\alpha_1|,|\alpha_2|\},$$

then $J = \mathbb{R} \times \{0\}$ is an M-summand, and thus an M-ideal in $\mathbb{R}^2$. A typical intersection of balls with J is indicated in figure (ii).

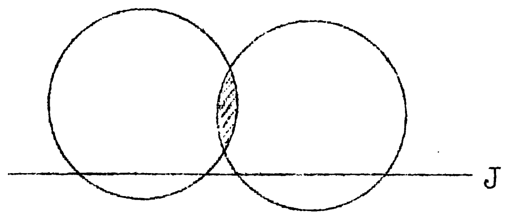


Figure (i)

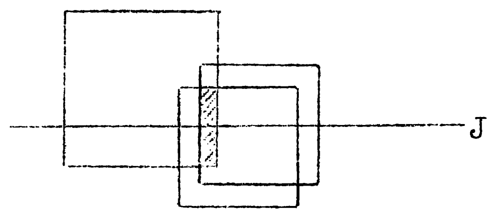


Figure (ii)

We show in Theorem 5.9 that the 2-ball property would not suffice in Theorem A.

The logical organization of the proof of Theorem A is outlined below. We note that some of the intermediate characterizations for M-ideals will undoubtedly be more useful than (vi) itself.

(i) the definition of M-ideals

$\Updownarrow$ Lemma 4.1

(ii) the linearity of certain envelopes of functions $\Leftarrow$

$\Downarrow$ Lemma 4.4 (see also Theorem 4.5)

(iii) a measure-theoretic property of M-ideals

$\Downarrow$ Lemma 5.2 (see also Theorem 5.4)

(iv) an extension theorem for M-ideals — Theorem 5.8

$\Downarrow$ Theorem 5.8

(v) the n-ball property for M-ideals

$\Updownarrow$ Theorem 5.9

(vi) the 3-ball property for M-ideals

A key step in the proof of (i) $\Longleftrightarrow$ (ii) is a geometrical characterization of the L-ideals in W . This is given in Proposition 3.1, and much of section 2 is devoted to the geometrical background needed for that result. The relevant result in section 2 is Theorem 2.9, which shows that closed subspaces, and more generally closed convex cones in W have large "complements". An important notion that is used in § 2 is that of "domination". This is a partial ordering of W that provides a substitute for the positive cone in the theory of ordered Banach spaces.

Section 3 is largely devoted to a stydy of the L-porjections in W , i.e., the projections on W determined by the L-summands in W . These were studied by Cunningham [C_1], who showed that they generate a commutative Banach subalgebra $\mathcal{C}(W)$ of the bounded operators on W . We call $\mathcal{C}(W)$ the Cunningham algebra of W . A characterization of the operators in $\mathcal{C}(W)$ is given in Theorem 3.12.

In section 6 we introduce the notion of a primitive M-ideal in V . The collection of such ideals together with a topology defined by a hull-kernel operation is completely analogous to the Jacobson structure space, and for (the self-adjoint part) of a C*-algebra it coincides with that space. We prove that it has the expected properties relative to ideals and quotients in Proposition 6.5.

In section 7 we introduce the centralizer $\mathcal{Z}(V)$ of V . This may be defined as the set of bounded operators T on V for which the adjoint T^* lies in $\mathcal{C}(W)$. It is shown the Theorem 7.6 that this is equivalent to having each extreme point of the dual ball an eigen-vector for T^* , and also to a strong boundedness property for T itself. The main result of section 7 is the following generalization of the Dauns-Hofmann Theorem for C*-algebras:

Theorem B. There is a natural isometry of the centralizer $\mathcal{Z}(V)$ onto the bounded structurally continuous functions on the primitive ideal space.

The precise statement of Theorem B is given in Theorem 7.7. The proof appears to be considerably more difficlut than that given in the ordered case.

In sections 8 and 9, it is shown how the theory of the preceding sections may be applied in certain special cases such as C*-algebras, ordered Banach spaces (including those associated with compact convex sets), and Lindenstrauss spaces (the pre-duals of Kakutani L-spaces). The main result of these sections is

Theorem C. If V is an Archimedean order unit space with a Banach pre-dual V , then the map $T \mapsto T^*$ carries $\mathcal{C}(U)$ onto $\mathcal{Z}(V)$.

This is presented in a somewhat more general form in Theorem 9.12.

Section 10 is devoted to the historical background of this paper, and some open problems.

The second author wishes to express his gratitude to Professors Dixmier and Choquet and their younger colleagues for the hospitality that they showed to him while on sabbatical leave at the University of Paris.

1. Preliminary notations and results.

Suppose that W is a vector-space. A non-empty subset C of W is a <u>cone</u> if $\lambda C \subseteq C$ for all $\lambda \geq 0$. If S is an arbitrary subset of W, the set

$$\text{cone } S = \bigcup_{\lambda \geq 0} \lambda S$$

is the smallest cone containing S. A cone C is said to be <u>proper</u> if $C \cap -C = \{0\}$, <u>symmetric</u> if $C = -C$, and <u>convex</u> if $C + C \subseteq C$ (this coincides with the usual notion of convexity). In particular, a cone is a subspace of W if and only if it is convex and symmetric. If C is a proper convex cone in W then we can define a (partial) ordering compatible with the linear structure by writing $p \leq q$ if $q - p \in C$. The restriction of this ordering to C is called the <u>intrinsic ordering</u> of C. An <u>ordered vector space</u> (W,C) is a vector space W ordered by a proper cone C.

Suppose that K is a convex subset of W. We shall say that a subset F of K is a <u>face</u> if F is convex and given $p, q \in K$ and $0 < \alpha < 1$ with $\alpha p + (1-\alpha)q \in F$, it follows that $p, q \in F$. A one point set $\{p\}$ is a face if and only if p is an extreme point. We denote the extreme points of K by $E(K)$. A face in a face F of K is a face in K, hence in particular $E(F) \subseteq E(K)$. An intersection of faces is a face, and if D is an arbitrary subset of K we let

$$\text{face}_K D = \bigcap \{F : F \text{ is a face and } F \supseteq K\} .$$

The following is well-known (and easily verified):

Lemma 1.1: If D is a convex subset of K, then $\mathrm{face}_K D$ consists of the $p \in K$ for which there exist $q \in K$ and $0 < \alpha < 1$ with $\alpha p + (1-\alpha)q \in D$.

Suppose that X is a topological space. We denote by $C(X)$, resp. $C^b(X)$, the space of real continuous, resp., bounded real continuous functions on X. The latter is an ordered Banach space with the uniform norm and the usual ordering. If X is compact Hausdorff, $C(X) = C^b(X)$, and the space of measures $M(X) = C(X)^*$ is given the dual norm and ordering. We let $M^+(X)$, resp., $P(X)$, be the $\mu \in M(X)$ with $0 \leq \mu$, resp., $\mu(1) = 1$ (i.e., $P(X)$ is the set of probability measures).

Suppose that W is a locally convex Hausdorff vector space, K is a compact convex set in W, and $p \in K$. We let $A(K)$ be the set of continuous affine functions on K, and $A_p(K)$ be the set of $a \in A(K)$ with $a(p) = 0$. $A(K)$ is a closed subspace of $C(K)$, and we give it the relative norm and topology.

Let us assume that $0 \in K$ and that W is the linear span of K. Each $a \in A_o(K)$ has a unique, but possibly discontinuous extension to a linear function on W (see e.g. [K_2; Lemma 4.1]), hence we shall refer to the members of $A_o(K)$ as <u>linear functions, continuous on</u> K. It is known (see e.g. [A_2; Cor.I.15]) that every $a \in A_o(K)$ can be uniformly approximated by continuously extendable linear functions; specifically $W^*|_K$ is <u>dense</u> in the norm of $A_o(K)$. If K is the closed unit ball of a dual Banach space W endowed with the w^*-topology, then $W^*|_K = A_o(K)$ by virtue of a known theorem going back to Banach (cf.e.g. [D.S; p.428]).

If $\mu \in M(K)$, we denote by $r(\mu)$ the resultant of μ in W, i.e. the weak integral of the indentity function $p \mapsto p$ on K. (Cf.e.g. [A_2; Prop.I.2.1] for an existence proof). It follows from the density of $W^*|_K$ in $A_0(K)$ that $r(\mu)$ is the unique point $p \in W$ such that

$$a(p) = \mu(a), \qquad \text{for all } a \in A_0(K) \tag{1.1}$$

If $\mu \in P(K)$, then $r(\mu) \in K$; and in this case $r(\mu)$ will be referred to as the barycenter of μ in K. If $a \in A(K)$ there exists a unique $a_0 \in A_0(K)$ and a scalar α with $a = a_0 + \alpha 1$. It follows from (1.1) that the barycenter of a probability measure $\mu \in P(K)$ is the unique point $p \in K$ such that

$$a(p) = \mu(a), \qquad \text{for all } a \in A(K) \tag{1.2}$$

We let $P_0(K)$ denote the $\mu \in P(K)$ with barycenter p.

We provide $M^+(K)$ with the usual dilation ordering: $\mu \prec \nu$ if $\mu(s) \leq \nu(s)$ for all convex $s \in C(K)$; and we say that a positive measure is maximal if it is maximal in that ordering. It follows from (1.2) and the convexity of both a and $-a$ for $a \in A(K)$, that if two probability measures μ, ν enjoy the relation $\mu \prec \nu$, then their barycenters must coincide.

If f is a real function on a set T, we define the subgraph of f, $\mathrm{Sub}_T f$ by

$$\mathrm{Sub}_T f = \{(p,\gamma) \in T \times \mathbb{R} : \gamma < f(p)\}, \tag{1.3}$$

and if B is a constant, we let

$$\mathrm{Sub}_T^B\, f = \{(p,\gamma) \in T \times \mathbb{R} : B \leq \gamma \leq f(p)\} \tag{1.4}$$

We give analogue definitions for the supergraph $\mathrm{Sup}_T f$ and $\mathrm{Sup}_T^B f$ (now truncating above by B). The graph of f is

$$\mathrm{Gr}_T f = \mathrm{Sub}_T f \cap \mathrm{Sup}_T f \ .$$

Returning to the compact convex set K, we give $K \times \mathbb{R}$ the product topology. We recall that f is a concave function if and only if $\mathrm{Sub}_K f$ is convex, and f is upper semi-continuous if and only if $\mathrm{Sub}_K f$ is closed. If f is bounded above, we define the upper envelope $\hat{f}$ of f by

$$\hat{f}(p) = \inf \{a \in A(K) : f \leq a\} \ ,$$

and we give an analogous definition for the lower envelope $\check{f}$ of f. One has the well-known results (see [Ph, Prop.3.1]) that if f is continuous, then

$$\hat{f}(p) = \sup \{\mu(f) : \mu \in P_p(K)\}$$

and that a measure $\mu \in P(K)$ is maximal if and only if $\mu(f) = \mu(\hat{f})$ for each $f \in C(K)$. Perhaps less well-known is the fact that these results are also valid for upper semi-continuous functions f (they are false for lower semi-continuous functions):

Lemma 1.2. Suppose that K is a compact convex subset of a locally convex Hausdorff space, and that f is a function on K which is bounded above. Then letting $\overline{\mathrm{co}}$ indicate closed convex hull,

(a) $\mathrm{Sub}_K \hat{f} = \overline{\mathrm{co}}\, [\mathrm{Sub}_K f]$

(b) If f is upper semi-continuous, then

$$\hat{f}(p) = \sup \{\mu(f) : \mu \in P_p(K)\} .$$

(c) If f is upper semi-continuous and convex,

$$\hat{f}(p) = \sup \{\mu(f) : \mu \in P_p(K), \mu \text{ maximal}\}.$$

Proof. For (a), see Prop. 2.1 and 2.2 of [Br]. Statement (b) may be found in [G.R., Prop.5.6]. Finally, (c) follows since $\mu \prec \nu$ implies $\mu(f) \le \nu(f)$ for every upper semi-continuous convex function f (see, for example, [A_2, p.22]).

Lemma 1.3: If f is upper semi-continuous, then $\mu(\hat{f}) = \mu(f)$ for all maximal measures $\mu \in P(K)$.

Proof: See [G.R., Cor.5.10], or [A_2, p.35].

We define $\mathcal{A}(K)$ to be the bounded Borel functions a on K which satisfy the barycentric calculus. By this we mean that if $p \in K$ and μ is a probability measure with resultant p , then $\mu(a) = a(p)$. Any such function is obviously affine. Conversely, if a is affine and continuous on K, then $a \in \mathcal{A}(K)$. More generally we say that a function f on K is quasi-continuous if for each compact set $D \subseteq K$, $f|_D$ has a dense set of points of continuity in D. Choquet proved that any quasi-continuous affine Borel function is bounded and satisfies the barycentric calculus, and he gave an example of an affine Borel function which does not [Ch 1](see also [A_2, § 2]). Upper and lower semi-continuous functions are quasi-continuous (see e.g. [B_2; Ch.IV,§ 6,no.2, ex.9]). The set of continuity points of a function is always a G_δ-set (consider oscillations), and an intersection of a finite number of dense G_δ-sets in a Baire space is dense. From this it follows that a finite convex combination of quasi-continuous functions is quasi-continuous. Thus we have the following:

<u>Lemma 1.4</u>: If an affine function a on K is a finite linear combination of upper (or lower) semi-continuous functions, then $a \in \mathfrak{A}(K)$, i.e., a is bounded, Borel, and satisfies the barycentric calculus.

We note that due to the Lebesgue Dominated Convergence Theorem, $\mathfrak{A}(K)$ is closed under dominated sequential convergence.

<u>Lemma 1.5</u>. Let f be a bounded Borel function on K satisfying the barycentric calculus. If the restriction of f to the closure Z of $E(K)$ is continuous, then f is continuous.

<u>Proof</u>. Consider the diagram

$$\begin{tikzcd} P(Z) \arrow[d, "r"'] \arrow[dr, "h"] & \\ K \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

where r is the resultant map $\mu \mapsto r(\mu)$, and h is the map $\mu \mapsto \mu(f)$. Since f satisfies the barycentric calculus, this diagram commutes. The function r is onto, and both r and h are continuous. For every closed subset F of $\mathbb{R}$ the set $f^{-1}(F) = r(h^{-1}(F))$ is compact, hence closed, and f is continuous.

2. Facial structure of balls.

Throughout this section we shall assume that W is a normed vector space with closed unit ball K. It should be noted, however, that one can extend the results of this section to a (possibly non-symmetric) convex body K with $0 \in \operatorname{int} K$ by replacing the norm by the Minkowski functional of K.

We denote by S the <u>surface</u> of K, i.e.

$$S = \{p \in W : \|p\| = 1\} ,$$

and we make the following simple observation which we state as a lemma for later references:

<u>Lemma 2.1</u>. Every proper face of K is contained in S, and any convex subset of S is contained in a proper face.

<u>Proof</u>: If Q is a face of K and $p \in Q-S$, then p is a proper convex combination of 0 and $p/\|p\|$. Thus $0 \in Q$. If $q \in K$ is arbitrary, $0 = \frac{1}{2}(q + (-q))$ hence $q \in Q$ and $K = Q$.

If D is a convex subset of S and $q \in \operatorname{face}_K D$, then from Lemma 1.1, $\alpha q + (1-\alpha)r \in D$ where $r \in K$ and $0 < \alpha < 1$. From $\|q\| \leq 1$, $\|r\| \leq 1$, and

$$1 = \|\alpha q + (1-\alpha)r\| \leq \alpha \|q\| + (1-\alpha)\|r\| ,$$

we conclude that $\|q\| = 1$, i.e., $\operatorname{face}_K D \subseteq S$.

<u>Corollary 2.2</u>: Any proper face in K is contained in a maximal proper face, and the maximal proper faces are closed.

<u>Proof:</u> Since an ascending union of proper faces is a proper

face (it doesn't contain 0), the first assertion is trivial. Due to the continuity of $p \mapsto \|q\|$, the closure of a maximal face is a convex subset of S , and we may apply Lemma 2.1.

We say that a cone C in W is <u>facial</u> if $C = \{0\}$ or C is the cone generated by a proper face F in K , i.e., $C = \text{cone } F$. Any proper cone is convex and proper. If $0 \neq p \in W$, then $p/\|p\| \in S$, and from Lemma 2.1 ,

$$C(p) = \text{cone } (\text{face}_K \{p/\|p\|\}) \tag{2.1}$$

is a facial cone (the smallest) containing p . If $p = 0$, we let $C(p) = \{0\}$. Turning to more than one element,

<u>Lemma 2.3</u>: Suppose that $p_1, \ldots, p_n \in W$. Then the following are equivalent:

(a) There is a facial cone containing $p_1, \ldots, p_n$.

(b) $p_1, \ldots, p_n \in C(p_1 + \ldots + p_n)$

(c) $\|\Sigma_{i=1}^n p_i\| = \Sigma_{i=1}^n \|p_i\|$

<u>Proof</u>: We may assume that $p_1, \ldots, p_n \neq 0$. Let $\alpha_i = \|p_i\| / \Sigma_{i=1}^n \|p_i\|$, and let f be the convex combination $q = \Sigma_{i=1}^n \alpha_i \, p_i/\|p_i\|$. The statement (c) is then just the assertion that $\|q\| = 1$.

(a) => (c). Let F be a proper face in K with $p_1, \ldots, p_n \in \text{cone } F$. Then $p_i/\|p_i\| \in F$ and since F is convex, $q \in F$ and $\|q\| = 1$.

(c) => (b). By assumption, $\|q\| = 1$ and from Lemma 2.1

$$p_i/\|p_i\| \in \text{face}_K \{q\} \subseteq S .$$

But $q = (\Sigma_{i=1}^n p_i)/\|\Sigma_{i=1}^n p_i\|$, hence

$$p_i \in \text{cone } (\text{face}_K \{q\}) = C(p_1+\ldots+ p_n) .$$

(b) => (a) is trivial, and the proof is complete.

If $p_1, \ldots, p_n$ satisfy any of the properties (a) - (c) , we say that they are <u>without</u> <u>cancellation</u>. If this is the case, $C(p_1+\ldots+ p_n)$ is the smallest facial cone containing $p_1, \ldots, p_n$. If p_1 and p_2 are without cancellation, we will write $p_1 | p_2$. From Lemma 2.3 we have:

<u>Corollary 2.4</u>: Suppose that $p,q \in X$. <u>The following are equivalent</u>:

(a) $p | q-p$.

(b) $0 \leq p \leq q \mod C(q)$.

(c) There is a facial cone C with $0 \leq p \leq q \mod C$.

(d) $0 \leq p \leq q \mod C$ for any facial cone C containing q.

Following [E_1] , we will write $p \prec q$ if any of the above conditions is valid. Thus $p \prec q$ if and only if

$$\|p\| + \|q-p\| = \|q\| \tag{2.2}$$

Since each $p \in W$ lies in a facial cone,

$$0 \prec p \quad \text{for all} \quad p \in W , \tag{2.3}$$

and since facial cones are proper,

$$p \prec q \quad \text{and} \quad -p \prec q \quad \text{implies that} \quad p = 0 . \tag{2.4}$$

<u>Lemma 2.5</u>: The relation $p \prec q$ is a partial ordering of W. Although it is not compatible with the linear structure, we have:

If $p_i \prec q_i$, $i = 1,\dots, n$, and the q_i are without cancellation, then the p_i are without cancellation and $\Sigma p_i \prec \Sigma q_i$, (2.5)

and if α is <u>any</u> scalar ,

$$p \prec q \text{ implies that } \alpha p \prec \alpha q . \tag{2.6}$$

<u>Proof</u>: We have that

$$p \prec q \text{ and } q \prec p \text{ implies } p = q , \tag{2.7}$$

since if C is a facial cone containing p and q , then $p \leq q$ and $q \leq p$ mod C implies $p = q$ (any facial cone is proper). Turning to transitivity,

$$p \prec q \prec r \text{ implies } p \prec r , \tag{2.8}$$

since if C is a facial cone containing r , we have from Corollary 2.4 (c) , $0 \leq p \leq q \leq r$ mod C , hence $0 \leq p \leq r$ mod C .

If $\prec$ were compatible with the linear ordering, we would have that if $p \neq 0$, $0 \prec -p$ implies $p \prec -p+p = 0$ hence since $0 \prec p$, $p = 0$, a contradiction. Suppose that p_i and q_i have the properties given in (2.5). Let C be a facial cone containing Σq_i . Then $0 \leq p_i < q_i$ mod C , the p_i are without cancellation (Lemma 2.3 (a)), and $0 \leq \Sigma p_i \leq \Sigma q_i$ mod C .

If C is a facial cone, then for any scalar α αC is also facial since if $\alpha < 0$, then $\alpha C = -C$.

Thus if $0 \leq p \leq q \mod C$, when $0 \leq \alpha p < \alpha q \mod \alpha C$, and (2.6) follows.

We note that for the more general theory of asymmetric convex bodies, one must assume that $\alpha \geq 0$ in (2.6) (this does not play an important role below).

Lemma 2.6: Suppose that $q \in W$. Then $C(q)$ consists of the $p \in W$ such that $p \prec \alpha q$ for some $\alpha > 0$.

Proof: If $p \prec \alpha q$, $p \in C(\alpha q) = C(q)$ (see Corollary 2.4). Conversely say that $p \in C(q)$. We may assume that $\|p\|=\|q\|=1$. Then $p \in \mathrm{face}_K \{q\}$, and from Lemma 1.1, there exist $r \in K$ and $0 < \beta < 1$ with $q = \beta p + (1-\beta)r$. It follows that $\beta^{-1}q = p + \beta^{-1}(1-\beta)r$. Since both p and $\beta^{-1}(1-\beta)r$ belong to $C(\beta^{-1}q) = C(q)$, it follows from Corollary 2.4 that $p \prec \beta^{-1}g$.

Lemma 2.7: Suppose that C is a cone in W . Then

(a) C is a union of facial cones if and only if

$p \prec q \in C$ implies $p \in C$.

(b) C is contained in a facial cone if and only if $p_1 | p_2$ for all p_1 and p_2 in C .

(c) C is a facial cone if and only if it satisfies the conditions of (a) and (b).

Proof: (a) C is a union of facial cones if and only if $C = \bigcup_{q \in C} C(q)$, hence (a) follows from Lemma 2.6.

(b) If C is contained in a facial cone, the second part of (b) follows from Lemma 2.3. Conversely, note that an increasing union of proper faces in K is again a proper face (it does not contain 0). It follows that an increasing union of facial cones is a facial cone. We have $C \subseteq \bigcup_{q \in C} C(q)$, and if $q_1 | q_2$, then from Lemma 2.3

$$C(q_1) + C(q_2) \subseteq C(q_1 + q_2) .$$

Thus the assumption $q_1 | q_2$ for $q_i \in C$ implies that the collection $\{C(q) : q \in C\}$ is directed upwards under inclusion, and C is contained in the facial cone $\bigcup_{q \in C} C(q)$.

(c) If C satisfies both (a) and (b) then $C = \bigcup_{q \in C} C(q)$ is a facial cone.

We say that a cone C in W is heriditary, resp. additive, if it satisfies the conditions of (a), resp. (b) of Lemma 2.7.

If C is a (not necessarily convex) cone in W, the complementary cone C' is defined to be the set of all $q \in W$ for which $C(q) \cap C = \{0\}$. From Lemma 2.6, these are just the points q such that if $p \prec q$ and $p \in C$, then $p = 0$. It should be noted that C' need not be proper or convex even if C has those properties. For example, if W is the plane and K is the closed unit disk, then for every $p \neq 0$, the facial cone $C(p)$ is the ray from 0 through p, and $C(p)'$ consists of the complement of $C(p)$ in W together with the point 0. On the other hand, C' is always hereditary, since if $p \prec q \in C'$ and $n \prec p$, $n \in C$, then $n \prec q$ and $n = 0$.

We now come to the major result of this section. It depends on a preliminary analogue of the "Monotone Convergence Theorem".

<u>Lemma 2.8</u>: Suppose that W is complete and that $\{p_\gamma\}$ is a net in W which is increasing in the ordering $\prec$, and bounded in norm. Then there is a $\prec$-least upper bound p for the set $\{p_\gamma\}$, and the net $\{p_\gamma\}$ converges in norm to p.

<u>Proof</u>: If $\gamma \leq \delta$, then $p_\gamma \prec p_\delta$, hence from (2.2)

$$\|p_\gamma\| = \|p_\delta\| - \|p_\gamma - p_\delta\| \leq \|p_\delta\| .$$

It follows that the net $\{\|p_\gamma\|\}$ is an increasing and bounded ned, hence it converges. From (2.2) if $\gamma < \delta, \delta'$,

$$\begin{aligned}\|p_\delta - p_{\delta'}\| &\leq \|p_\delta - p_\gamma\| + \|p_{\delta'} - p_\gamma\| \\ &= \|p_\delta\| + \|p_{\delta'}\| - 2\|p_\gamma\| .\end{aligned}$$

It follows that $\{p_\gamma\}$ is Cauchy and we may let $p = \lim p_\gamma$. If $\gamma \leq \delta$,

$$\|p_\delta\| = \|p_\gamma\| + \|p_\delta - p_\gamma\| ,$$

hence taking the limit over δ,

$$\|p\| = \|p_\gamma\| + \|p - p_\gamma\| ,$$

i.e., $p \prec p_\gamma$. If $p_\gamma \prec q$ for all γ, then

$$\|q\| = \|p_\gamma\| + \|q - p_\gamma\| ,$$

and taking the limit over γ, it follows that $q \succ p$.

<u>Theorem 2.9</u>: Suppose that W is complete and that C is a norm-closed convex (not necessarily proper) cone in W. Then every $p \in W$ admits a decomposition

$$p = q + r , \quad q | r \tag{2.8}$$

where $q \in C$ and $r \in C'$. Given q_o with $q_o \precsim p$, q may be chosen with $q_o \precsim q$.

Proof: Let $A = \{u \in C : u \precsim p\}$ (note that one always has $0 \in A$), and let B be a maximal subset of A that is totally ordered by $\precsim$. If $q_o \in C$ and $q_o \precsim p$ we may assume that $q_o \in B$. If $u \in C$, $\|u\| \leq \|p\|$, hence from Lemma 2.8, B has a $\precsim$-least upper bound q in the closure of B. It follows that $q \in C$ and $q \precsim p$, hence $q \in A$. But $u \precsim q$ for all $u \in B$, hence from the maximality of B, $q \in B$, and q is a maximal element of A. Letting $r = p-q$, $q|r$ and we claim that $r \in C'$. We must show that if $s \in C$ and $s \precsim r$, then $s = 0$. Since also $q \precsim q$ and $q|r$ we have from (2.5) that $s|q$, i.e. $q \precsim q+s$, and $q+s \precsim q+r = p$. Since C is convex $q+s \in C$, hence $q+s \in A$, or since q is maximal, $s = 0$.

If K is a convex subset of a vector space and D is a subset of K, the complementary set $D^c = D^c_K$ is the union of all faces in K disjoint from D.

Corollary 2.10. Suppose that W is a Banach space, K is the closed unit ball of W and that D is a closed convex subset of the surface S. Then every point in S is a convex combination of a point in D and a point in D^c.

Proof: $C = \text{cone } D$ is a closed convex cone, for suppose that $\{p_n\}$ is a sequence in C converging to p. If $p = 0$ there is nothing to prove. If $p \neq 0$, $\|p\| = \lim \|p_n\|$ implies that $p/\|p\| = \lim p_n/\|p_n\|$, or since $p_n\|p_n\| \in D$, $p/\|p\| \in D$ and

$p \in C$. Given $p \in S$, we have from Theorem 2.9 that there exist $q \in C$ and $r \in C'$ with $p = q+r$ and $1 = \|p\| = \|q\| + \|r\|$, If q or $r = 0$ we are done. If neither is 0 ,

$$p = \|q\|(q/\|q\|) + \|r\|(r/\|r\|)$$

where $q/\|q\| \in C \cap S = D$. Since $C(r) \cap C = \{0\}$, $\mathrm{face}_K \{r/\|r\|\} \cap D = \emptyset$, and $r/\|r\| \in D^c$.

<u>Corollary 2.11</u>: Assume the hypotheses of Corollary 2.10. Then $D^{cc} \subseteq D$, and $D^{cc} = D$ if and only if D is a face .

<u>Proof:</u> We have from Corollary 2.10 that a face in K must be contained in D or D^c , or it must intersect both. The assertions immediately follow from this.

Suppose that H is a compact convex subset of a locally convex space. There is a natural map δ of H into $A(H)^*$ defined by $\delta(p)(a) = a(p)$. If $A(H)^*$ is given the weak* topology, δ is an affine weak* homeomorphism onto the state space of $A(H)$

$$SA(H) = \{p \in A(H)^* : 0 \leq p , \quad p(1) = 1\} .$$

Identifying H with $SA(H)$, the <u>intrinsic norm topology</u> on H is the relative topology on H defined by the norm in $A(H)^*$.

<u>Proposition 2.12</u>: Suppose that H is a compact convex subset of a locally convex Hausdorff space, and that D is a proper, convex subset of H closed in the intrinsic norm topology. Then $D^c \neq \emptyset$ and each point in H is a convex combination of a point in D and a point in D^c .

<u>Proof:</u> Let K be the unit ball of the Banach space $A(K)^*$. Then H is identified with the points in K at which the function $1 \in A(H)$ assumes its maximum value, i.e., the scalar 1. It follows that H is a weak* , hence norm closed face in K , and since it is proper, H is contained in the surface S of K . From Corollary 2.10, if $p \in H$, then there exist $0 \leq \alpha < 1$ $q \in D$, and $r \in D_K^c$ with $p = \alpha q + (1-\alpha)r$. If $0 < \alpha < 1$ then since H is a face in K , $r \in H$, and $\text{face}_H \{r\} \subseteq \text{face}_K \{r\}$ is disjoint from D , i.e., $r \in D_H^c$. In particular letting p be a point in $H-D$, we see that $D_H^c \neq \emptyset$. If $\alpha = 1$, we may replace r by any point in D_H^c . Finally $\alpha \neq 0$ since $D \cap D_K^c = \emptyset$.

<u>Remark 2.13:</u> Corollary 2.13 has long been known for K a simplex and D a compact face [A_1], and it has recently been extended to the case of a compact convex set K and a compact face D [A_2, Prop.II. 6.5], and by Ellis also to a compact simplex K and a norm closed face D [E_1, p. 3].

3. L-ideals and the Cunningham algebra.

In this section W shall be a fixed Banach space with closed unit ball K. Since the linear subspaces of W are just the symmetric convex cones, our discussion of cones is relevant. In particular, we have the notions of a <u>hereditary subspace</u> of W, and the <u>complementary cone</u> N' of a subspace N. N' must be a symmetric hereditary cone since from the symmetry of K, $C(-p) = -C(p)$.

Following Cunningham [C_1], we define an <u>L-projection</u> e on W to be a linear map of W into itself such that

L_1 : e is a projection, i.e., $e^2 = e$,

L_2 : $\|p\| = \|ep\| + \|p-ep\|$ for all $p \in W$.

Any L-projection e is bounded, and if I is the identity operator, then $I-e$ is also an L-projection. The statement L_2 may be reformulated

L_2' : $ep \prec p$ for all $p \in W$.

If W_1 and W_2 are Banach spaces and we give the direct sum $W = W_1 \oplus W_2$ the norm

$$\|(w_1,w_2)\| = \|w_1\| + \|w_2\| ,$$

then the operator e on W defined by

$$e(p_1,p_2) = (p_1,0)$$

is an L-projection. It is readily verified that after the usual identifications, all L-projections are of this form.

Suppose that e is an L-projection on W. If C is a hereditary cone in W, we have from L_2' that $eC \subseteq C$.

If $p \prec q$, then there is a facial cone C with $p \in C$ and $q-p \in C$. Thus $ep \in C$ and $eq-ep \in C$, which imply that $ep \prec eq$. We conclude that

$$p \prec q \text{ implies that } ep \prec eq. \tag{3.1}$$

In particular, the range eW of an L-projection e must be a hereditary subspace of W since if $p \prec q \in eW$, then $(I-e)p \prec (I-e)q = 0$. It follows that if f is another L-projection, then f leaves eW fixed, i.e., $fe = efe$. Replacing e by $I-e$, $ef = efe$, and we have Cunningham's result that <u>L-projections</u> <u>communte</u> [C_1, Lemma 2.2]. We say that a subspace of W is an <u>L-ideal</u> or <u>L'summand</u> if it is the range of an L-projection.

By the above remarks <u>any</u> <u>L-ideal is an hereditary subspace</u> (The opposite is false, as can be seen by simple examples in the plane).

<u>Proposition 3.1</u>: Suppose that N is a closed subspace of a Banach space W. Then the following are equivalent:

(a) $co(N') \cap N = \{0\}$

(b) N' is convex

(c) N' is a linear subspace of W, and W is the direct sum of N and N',

(d) N is an L-ideal.

Suppose that N is an L-ideal, say $N = eW$ where e is an L-projection. Then $N' = (I-e)W$, and e is the only L-projection with range N.

<u>Proof</u>: (a) => (b). Since N' is a symmetric cone, $co(N')$ is a convex symmetric cone, i.e., a subspace of W. If N'

is not convex, we may find $p_1, \ldots, p_n \in N'$ with $\sum_{i=1}^{n} p_i \in co(N') \setminus N'$. From Theorem 2.9, $\sum_{i=1}^{n} p_i = q+r$ where $0 \neq q \in N$ and $r \in N'$. Thus

$$q = \sum_{i=1}^{n} p_i - r \in (co\ N') \cap N$$

and $co(N') \cap N \neq \{0\}$.

(b) => (c) and (d). From (b), N' is a convex symmetric cone, and thus a subspace of W. From Theorem 2.9, for each $p \in W$ there exist $q \in N$ and $r \in N'$ with $p = q+r$ and $q|r$. On the other hand from the definition of N', $N \cap N' = \{0\}$, hence W is the direct sum of N and N'. Let e be the projection of W on N along N', i.e., $ep = q$. Since $q|r$, $ep \prec p$ and e is an L-projection.

(c) => (a) is trivial.

(d) => (a). It suffices to prove that $N' = (I-e)W$. If $r \in (I-e)W$ suppose that $s \in eW$ and $s \prec r$. From (3.1), $s = es \quad er = 0$, hence $s = 0$, and so $r \in (eW)'$. Conversely suppose that $r \in (eW)'$. Since the latter is hereditary, it is left invariant by e, and $er \in (eW)' \cap eW = \{0\}$. Thus $r = (I-e)r$.

The unicity of e is a consequence of the relation $(I-e)W = N'$, since any projection f on W is determined by the subspaces fW and $(I-f)W$.

Let $\mathcal{B}(W)$ be the Banach algebra of bounded linear operators with the uniform norm $\| \ \|$. The <u>Cunningham algebra</u> $\mathcal{C}(W)$ is the Banach subalgebra of $\mathcal{B}(W)$ generated by the L-projections. Since the L-projections commute, $\mathcal{C}(W)$ is a commutative Banach algebra with identity. Let Ω be the spectrum of $\mathscr{C}(W)$. Cunningham proved that Ω is hyperstonean (see below for definition) and that the Gelfand transform is an isometric isomorphism of $\mathscr{C}(W)$ onto $C(\Omega)$ [C_1] . Our next object is to give an intrinsic characterization of the operators in the Cunningham algebra (see Theorem 3.12). Cunningham's results will follow from our approach.

Suppose that $S_1, \ldots, S_n$ are linear operators on W . We say that they are <u>without</u> <u>cancellation</u> if

$$\|S_1 p + \ldots + S_n p\| = \|S_1 p\| + \ldots + \|S_n p\|$$

for each $p \in W$. We write $S|T$ if S and T are without cancellation, and $S \prec T$ if $S|(T-S)$. Clearly $S \prec T$ if and only if $Sp \prec Tp$ for all $p \in W$. With these definitions, it is immediate that $\prec$ is a partial ordering on the linear operators satisfying (2.2) - (2.6). In addition, if $\alpha, \beta \geq 0$, then $\alpha p | \beta p$ for all $p \in W$, hence

$$\alpha I | \beta I \quad \text{for all} \quad \alpha, \beta \geq 0 . \tag{3.2}$$

Let $\mathscr{D}^+$ be the set of linear operators S on W for which $S \prec \alpha I$ for some scalar $\alpha \geq 0$, i.e.,

$$\|Sp\| + \|\alpha p - Sp\| = \alpha \|p\| \tag{3.3}$$

for all $p \in W$, and let $\mathscr{D} = \mathscr{D}^+ - \mathscr{D}^+$. As was the case for L-projections (which we note lie in $\mathscr{D}^+$), if $S \in \mathscr{D}^+$ and C is a hereditary cone, then $S(C) \subseteq C$. In particular, if $p \prec q$, then let C be a facial cone with $p \in C$ and $q-p \in C$. Then $Sp \in C$ and $Sq - Sp \in C$, hence $Sp \prec Sq$. We conclude that

$$S \in \mathscr{D}^+ \quad \text{and} \quad p \prec q \quad \text{implies} \quad Sp \prec Sq. \tag{3.4}$$

<u>Lemma 3.2</u>: $\mathcal{D}^+$ is a proper convex cone of operators in $\mathcal{B}(W)$, and it is closed under composition.

<u>Proof</u>: From (3.3) it is evident that if $s \prec \alpha I$, then $\|S\| \leq \alpha$. If $S \prec \alpha I$ and $\beta \geq 0$, we have from (2.6) that $\beta S \prec \beta\alpha I$. If $T \prec \beta I$, $\beta \geq 0$ then from (3.2) $\alpha I | \beta I$, hence we have from (2.5) that $S + T \prec (\alpha+\beta)I$. Suppose that $S, - S \in \mathcal{D}^+$ and choose $\alpha, \beta \geq 0$ with $S \prec \alpha I$, $- S \prec \beta I$. Letting $\gamma = \max\{\alpha,\beta\}$, we have $\alpha I \prec \gamma I$ and $\beta I \prec \gamma I$, and $S = 0$ is a consequence of (2.4). If $S \prec \alpha I$ and $T \prec \beta I$, then from (3.4), if $p \in W$,

$$STp \prec S\beta p \prec \alpha\beta p ,$$

i.e. $ST \in \mathcal{D}^+$.

It follows from Lemma 3.2 that $\mathcal{D}^+$ defines a partial ordering $\leq$ on $\mathcal{D}$, and composition defines a bilinear product on $\mathcal{D}$.

<u>Lemma 3.3</u>: The partial orderings $\prec$ and $\leq$ coincide on $\mathcal{D}^+$.

<u>Proof</u>: Suppose that S and T are in $\mathcal{D}^+$. If $S \prec T \prec \beta I$, $\beta \geq 0$, then $T - S \prec T$ implies $T - S \prec \beta I$, i.e., $T - S \in \mathcal{D}^+$ and $S \leq T$. Conversely, if $S \leq T$, $T - S \in \mathcal{D}^+$, and there exists a $\beta \geq 0$ with $T - S \leq \beta I$. Assuming that $S \leq \alpha I$, $\alpha \geq 0$, we have $\alpha I | \beta I$ (see (3.2)) and from (2.5), $S | T - S$, i.e., $S \prec T$.

An element I of a partially ordered vector space E is said to be an <u>order unit</u> if for each $S \in E$ there is an $\alpha \geq 0$ with $-\alpha I \leq S \leq \alpha I$. I is <u>Archimedean</u> if $S \leq \epsilon I$ for all $\epsilon > 0$ implies that $S \leq 0$.

Lemma 3.4. I is an Archimedean order unit for $\mathcal{D}$.

Proof. If $S \prec \alpha I$, then $S \in \mathcal{D}^+$, and from Lemma 3.3, $0 \leq S \leq \alpha I$. If in addition $T \prec \beta I$ and $\gamma = \max\{\alpha,\beta\}$, then $-\gamma I \leq S - T \leq \gamma I$, hence I is an order unit for $\mathcal{D}$. To prove I is Archimedean, we suppose that $S - T \leq \epsilon I$ where $0 \leq S,T$. Then $0 \leq S \leq \epsilon T + T$, hence from Lemma 3.3, $S \prec \epsilon I + T$, and for all $p \in W$,

$$\|Sp\| + \|(T-S)p + \epsilon p\| = \|Tp + \epsilon p\| .$$

If $\epsilon > 0$ is arbitrary, we conclude

$$\|Sp\| + \|(T-S)p\| = \|Tp\| ,$$

i.e., $S \prec T$, $S \leq T$, and $S - T \leq 0$.

If I is an Archimedean order unit in a partially ordered vector space E, we may define a norm $\| \|_I$ on E by

$\|S\|_I = \inf\{\alpha: -\alpha I \leq S \leq \alpha I\}$, where the infimum is actually attained (see [K_1]).

Lemma 3.5. $\mathcal{D}$ consists of all operators $S \in \mathcal{B}(W)$ for which there exists an $\alpha \geq 0$ such that

$$\|Sp + \alpha p\| + \|Sp - \alpha p\| = 2\alpha\|p\| \quad \text{for all } p \in W . \tag{3.5}$$

If $S \in \mathcal{D}$, then (3.5) is valid for every $\alpha \geq \|S\|_I$, and $\alpha = \|S\|_I$ is the smallest scalar for which (3.5) holds.

Proof. 1) Assume first that $S \in \mathcal{D}$. Then there is an $\alpha \geq 0$ such that

$$-\alpha I \leq S \leq \alpha I \tag{3.6}$$

Then $0 \leq S + \alpha I \leq 2\alpha I$, and from Lemma 3.3 $S + \alpha I \prec 2\alpha I$, which is equivalent to (3.6). From the definition of $\|S\|_I$, the relation (3.6) is valid for every $\alpha \geq \|S\|_I$, and $\alpha = \|S\|_I$ is the smallest scalar for which (3.6) holds. Hence the same statements are true with (3.5) in place of (3.6).

2) Assume next that S satisfies (3.5), or equivalently that $S + \alpha I \prec 2\alpha I$. Writing $S_1 = S + \alpha I$ and $S_2 = \alpha I$, we obtain $S = S_1 - S_2$ where $S_1 \in \mathscr{D}^+$ and $S_2 \in \mathscr{D}^+$. This completes the proof.

If $S \in \mathscr{D}$ and $\alpha \geq \|S\|_I$, then by (3.5):

$$
\begin{aligned}
2\|Sp\| &= \|Sp - (-Sp)\| \\
&\leq \|Sp - \alpha p\| + \|\alpha p - (-Sp)\| = 2\alpha\|p\| ,
\end{aligned}
$$

hence $\|S\| \leq \alpha$, and in general,

$$
\|S\| \leq \|S\|_I . \tag{3.7}
$$

In the proof of Lemma 3.8 we will see that $\|S\| = \|S\|_I$.

<u>Lemma 3.6:</u> $\mathscr{D}$ is complete in the norm $\| \ \|_I$.

<u>Proof</u>: Suppose that $S_n \in \mathscr{D}$ is Cauchy in the norm $\| \ \|_I$. From (3.7) it is Cauchy in $\mathscr{B}(W)$, hence we may assume that S_n converges uniformly to $S \in \mathscr{B}(W)$. Since S_n is $\| \ \|_I$ - Cauchy, it is bounded, i.e., there is a constant α with $\|S_n\|_I \leq \alpha$ for all n . From (3.5), if $p \in W$,

$$
\|S_n p + \alpha p\| + \|S_n p - \alpha p\| = 2\alpha\|p\|
$$

hence

$$
\|Sp + \alpha p\| + \|Sp - \alpha p\| = 2\alpha\|p\| ,
$$

i.e., $S + \alpha I \prec 2\alpha I$ and $S \in \mathscr{D}$. Given $\varepsilon > 0$, let n_o be such that $n,m \geq n_o$ implies $\|S_n - S_m\|_I \leq \epsilon$. Then

$$\|(S_n - S_m)p + \epsilon p\| + \|(S_n - S_m)p - \epsilon p\| = 2\epsilon\|p\| ,$$

hence

$$\|(S_n - S)p + \epsilon p\| + \|(S_n - S) - \epsilon p\| = 2\epsilon\|p\| ,$$

$\|S_n - S\|_I \leq \epsilon$, and S_n converges to S in the norm $\| \ \|_I$.

We shall need the following version of Stone's representation theorem for ordered algebras which was proved by Kadison in [K_1, p.7-9]:

<u>Lemma 3.7</u>: Suppose that E is a (partially) ordered vector space with an Archimedean unit I , and that E is complete in the norm $\|-\|_I$. Assume that E has a bilinear multiplication for which I is an identity, and that $ST \geq 0$ whenever $S \geq 0$ and $T \geq 0$. Then E is a commutative (real) Banach algebra and the spectrum Ω of E consists of all extreme points of the w^*-compact convex set Σ of states on E , i.e. of positive linear functionals p with $p(I) = 1$. Moreover, the Gelfand transform is an isometric order - and algebra - isomorphism of E onto $C(\Omega)$.

<u>Lemma 3.8</u>: $\mathscr{D}$ is a commutative Banach algebra, and the norms $\| \ \|$ and $\| \ \|_I$ coincide on $\mathscr{D}$. If Ω is the spectrum of , then the Gelfand transform is an isometric algebraic and order isomorphism of $\mathscr{D}$ onto $C(\Omega)$.

<u>Proof</u>: We have shown that with the norm $\| \ \|_I$ and the composition product, $\mathscr{C}(W)$ satisfies the conditions of Lemma 3.7. From (3.7), we may therefore regard the identity map as a norm-decreasing isomorphism of $C(\Omega)$ into $\mathscr{B}(W)$. Since

Kaplansky has shown that the supremum norm on $C(\Omega)$ is minimal among the submultiplicative norms on $C(\Omega)$, this map is an isometry [Kap_1, Th.6.2]. (We are indebted to J. Lindenstrauss for this reference).

We shall use Lemma 3.8 to identify $\mathcal{D}$ and $C(\Omega)$.

<u>Corollary 3.9</u>: The L-projections are just the idempotents in $\mathcal{D}$.

<u>Proof</u>: From L_2', the L-projections are the idempotent linear maps e satisfying $e \prec I$, hence they lie in $\mathcal{D}$. Conversely, if e is an idempotent in $C(\Omega)$, then $e = \chi_G$, where G is an open and closed set in Ω. Since $0 \leq \chi_G \leq 1$, $e = \chi_G \prec I$ (Lemma 3.3), and e is an L-projection.

We define the <u>strong topology</u> on $\mathcal{D}$ to be the weakest topology in which the functions $S \mapsto \|Sp\|$ are continuous for all $p \in W$. The following is immediate from Lemmas 2.8 and 3.3.

<u>Lemma 3.10</u>: If $\{S_\gamma\}$ is an increasing net in $\mathcal{D}$ with $0 \leq S_\gamma \leq T$ for some $T \in \mathcal{D}$, then there is a least upper bound S for the set $\{S_\gamma\}$. In addition the net $\{S_\gamma\}$ converges strongly to S.

We recall that if Ω is a compact Hausdorff space, then $C(\Omega)$ is <u>boundedly complete</u> if given an increasing net $\{f_\gamma\}$ in $C(\Omega)$ with $0 \leq f_\gamma \leq g$, $g \in C(\Omega)$, there is a least upper bound $f = \vee f_\gamma$ in $C(\Omega)$ for the family f_γ (this need not be the point-wise supremum). $C(\Omega)$ is boundedly complete if and only if Ω is <u>extremally disconnected</u>, i.e., the closure of each open set is open (see [G.J., § 3 N]).

It follows that

Corollary 3.11: The spectrum Ω of $\mathcal{D}$ is extremally disconnected.

Theorem 3.12: Let W be a real Banach space. Then the Cunningham algebra $\mathcal{C}(W)$ coincides with $\mathcal{D}$, i.e., with the algebra of linear operators S on W which satisfy (3.5) for some $\alpha \geq 0$.

Proof: Since $\mathcal{D}$ is a Banach subalgebra of $\mathcal{B}(W)$ containing the L-projections (Lemma 3.8 and Corollary 3.9), $\mathcal{C}(W) \subseteq \mathcal{D}$. The algebra $\mathcal{D}_0 \subseteq \mathcal{D}$ generated by the projections consists of finite sums of the form $\sum_i c_i \chi_{G_i}$ where the G_i form a disjoint partition of Ω into open closed sets. Since Ω is an extremally disconnected compact Hausdorff space (Corollary 3.11), $\mathcal{D}_0$ separates points in Ω, and from the Stone-Weierstrass Theorem, $\mathcal{D}_0$ is dense in $C(\Omega) = \mathcal{D}$. Since $\mathcal{D}_0 \subseteq \mathcal{C}(W)$, we conclude that $\mathcal{D} = \mathcal{C}(W)$.

Suppose that Ω is an extremally disconnected compact Hausdorff space. A measure μ on Ω is said to be normal if for each increasing, uniformly bounded net $\{f_\gamma\}$ in $C(\Omega)$, $\mu(\vee f_\gamma) = \sup \mu(f_\gamma)$. Ω is hyperstonean if there is a famlily of normal measures on Ω which distinguishes functions in $C(\Omega)$. The following was proved by Cunningham:

Proposition 3.13: Let W be a real Banach space. Then the spectrum Ω of the Cunningham algebra $\mathcal{C}(W)$ is hyperstonean.

Proof: From Corollary 3.11, Ω is a compact extremally disconnected. If $p \in W$, then $S \mapsto \|Sp\|$ is additive and positively homogeneous on $\mathcal{C}(W)^+$. Thus it extends to a positive linear function μ_p on $\mathscr{C}(W)$. Let $0 \leq S_\gamma$ be an increasing net with least upper bound S. From Lemma 3.10, S_γ converges strongly to S. In particular $\mu_p(S_\gamma) = \|S_\gamma p\|$ converges to $\mu_p(S) = \|Sp\|$.

Suppose that $S, T \in \mathscr{C}(W)^+$ and $S \neq T$. We may assume that for some $\omega_0 \in \Omega$, $S(\omega_0) > T(\omega_0) + \epsilon$, where $\epsilon > 0$. The set $G = \{\omega : S(\omega) > T(\omega) + \epsilon\}$ is open, hence its closure $\bar{G}$ is open-closed and $e = \chi_{\bar{G}}$ is an L-projection with $0 \leq Te \leq (T+\epsilon)e \leq Se$. Thus $Te \prec (T+\epsilon)e \prec Te$, and selecting $0 \neq p \in eW$, we get

$$\mu_p(S) = \|Sp\| \geq \|T+\epsilon)p\| = \|Tp\| + \epsilon\|p\| ,$$

and $\mu_p(S) > \|Tp\| = \mu_p(T)$. It follows that the functionals μ_p, $p \in \Omega$, distinguish operators in $\mathscr{C}(W)^+$ and thus in $\mathscr{C}(W)$.

Using the lattice operations in $\mathscr{C}(W) = C(\Omega)$, we have for L-projections e and f that $e \wedge f = ef$ and $e \vee f = e + f - ef$. It follows that for the corresponding L-ideals we have

$$(e \wedge f)W = eW \cap fW \tag{3.8}$$

$$(e \vee f)W = eW + fW , \tag{3.9}$$

i.e., the intersection and sum of two L-ideals is again an L-ideal. If $\{e_\gamma\}$ is an increasing net of L-projections, it is bounded by I, and we have an L-projection $e = \bigvee_\gamma e_\gamma$. Since $e_\gamma p$ converges in norm to ep for each p (Lemma 3.10), we have that

$$(\bigvee_\gamma e_\gamma)W = (\bigcup_\gamma e_\gamma W)^-, \tag{3.10}$$

where the bar indicates norm-closure. Extending (3.9) to finite unions and sums we conclude that for any collection of L-projections e_γ,

$$(\bigvee_\gamma w_\gamma)W = (\sum_\gamma e_\gamma W)^- \tag{3.11}$$

where on the right we consider sums of finitely many elements. A similar argument shows that if e_γ is a decreasing net, then

$$(\bigwedge_\gamma e_\gamma)W = \bigcap_\gamma e_\gamma W \ , \tag{3.12}$$

and using (3.8), this extends to arbitrary collections $\{e_\gamma\}$. We conclude

Proposition 3.14: If $\{N_\gamma\}_{\gamma \in \Gamma}$ are L-ideals in W, then

a) $\bigcap N_\gamma$ is an L-ideal in W.

b) $(\Sigma N_\gamma)^-$ is an L-ideal in W (bar indicates norm-closure)

c) If Γ is finite, then ΣN_γ is an L-ideal in W.

Turning to ideals and quotients,

Proposition 3.15: Suppose that W is a real Banach space, and that e is an L-projection in W. Let N be the L-ideal e W, and if $N \neq W$, let ϑ be the quotient map of W onto W/N. If $e \neq 0$,

(a) The map $T \mapsto Te$ (the composition of T and e) is an isometric isomorphism of $\mathscr{C}(N)$ onto $\mathscr{C}(W)e$.

(b) The L-ideals in N are just the L-ideals of W that are contained in N.

If $e \neq I$,

(c) ϑ restricts to an isometry of $(I-e)W$ onto W/N.

(d) The ϑ-images of L-ideals are L-ideals, and the inverse images of L-ideals in W/N are the L-ideals in W containing N.

Proof. (a) It is evident that $T \mapsto Te$ is an isometric isomorphism of $\mathscr{C}(N)$ into $\mathscr{B}(W)$. It is also clear that if S and T are operators on N with $S \prec T$ then $Se \prec Te$ on W. If $T \in \mathscr{C}(N)$ let $\alpha = \|T\|$. Then letting I_N (resp., I) denote the identity map on N (resp., W), $T + \alpha I_N \prec 2\alpha I_N$ (see Lemmas 3.5 and 3.8). Composing with e,

$$Te + \alpha e \prec 2\alpha e \prec 2\alpha I,$$

hence $Te \in \mathscr{C}(W)$. If $S \in \mathscr{C}(W)e$, the restriction $S_1 = S|_N$ is an element of $\mathscr{C}(N)$ with $S = S_1 e$, hence the map is onto.

(b) If f is an L-projection in N, then from (a), fe is a projection in $\mathscr{C}(N)$, i.e., an L-projection, and $fN = feW$ is an L-ideal in W. Conversely if f is an L-projection in W with $fW \subseteq N = eW$, then $f_1 = f|_N$ is an L-projection on N with $f = f_1 e$, and $fW = f_1 W$ is an L-ideal in N.

(c) If $p \in (I-e)W$ and $q \in N = eW$, then

$$\|p + q\| = \|p\| + \|q\| \geq \|p\|$$

and

$$\|\vartheta(p)\| = \inf\{\|p + q\| : q \in N\} = \|p\|$$

For any $r \in W$, $\vartheta(r) = \vartheta((I-e)r)$, hence ϑ maps $(I-e)W$ onto W/N.

(d) From (c), $\vartheta_1 = \vartheta|_{(I-e)W}$ is an isometry of (I-e)W onto W/N, and this defines a one-to-one correspondence between L-ideals in these spaces. If $N_o = fW$ is an L-ideal in W, then

$$(I-e)N_o = [(I-e)f]W$$

is an L-ideal in W since (I-e)f is an L-projection. From (b), it is an L-ideal in (I-e)W, and

$$\vartheta(N_o) = \vartheta_1(I-e)N_o)$$

is an L-ideal in W/N. If $N_o \supseteq N$, then $N_o = \vartheta^{-1}\vartheta(N)$. Conversely if N_1 is an L-ideal in W/N, $\vartheta_1{}^{-1}(N_1)$ is an L-ideal in (I-e)W, and thus in W, and from (c) of Proposition 3.14,

$$\vartheta^{-1}(N_o) = \vartheta_1^{-1}(N_o) + N$$

is an L-ideal in W containing N.

We conclude this section with

<u>Proposition 3.16</u>: If N is a hereditary subspace of W, then the set of extreme points of $N \cap K$ is given by the formula:

$$E(N \cap K) = N \cap E(K) \tag{3.13}$$

If N_1 and N_2 are L-ideals in W, then

$$(N_1 + N_2) \cap K = \mathrm{co}[(N_1 \cap K) \cup (N_2 \cap K)] \tag{3.14}$$

and

$$(N_1 + N_2) \cap E(K) = [N_1 \cap E(K)] \cup [N_2 \cap E(K)]. \tag{3.15}$$

<u>Proof:</u> 1) The inclusion $N \cap E(K) \subseteq E(N \cap K)$ is obvious for any subset N of W. If N is a hereditary subspace of W, it is a union of facial cones, hence $N \cap S$ is a union of faces in K. It follows that if $p \in E(N \cap K) \subset S$, there is a face F of K with $p \in F \subseteq N \cap K$. Thus $p \in E(F) \subseteq E(K)$, and we have (3.13).

2) Given L-ideals N_1 and N_2, let e_1 and e_2 be the L-projections with $N_i = e_iW$. If p is in

$$(N_1 + N_2) \cap K = (e_1 \vee e_2)W \cap K,$$

(see (3.9)), then

$$p = (e_1 \vee e_2)p = p_1 + p_2$$

where $p_1 = e_1p$ and $p_2 = (e_2 - e_1e_2)p$. Since $p_1 \dashv p$, we have $p_1 | p_2$ and

$$\|p_1\| + \|p_2\| = \|p\| \leq 1.$$

Assuming that $p_i \neq 0$ (otherwise delete the term)

$$p = \|p_1\| \frac{p_1}{\|p_1\|} + \|p_2\| \frac{p_2}{\|p_2\|} + (1 - \|p\|)0 \ ,$$

hence

$$(N_1 + N_2) \cap K \subseteq co[(N_1 \cap K) \cup (N_2 \cap K)].$$

Since the reverse inclusion is trivial, (3.14) follows.

3) From (3.14) it is immediate that

$$E[(N_1 + N_2) \cap K] \subseteq E(N_1 \cap K) \cup E(N_2 \cap K) \ . \qquad (3.16)$$

It is easily seen that N_1 is an L-ideal, and hence a hereditary subspace of $N_1 + n_2$. Since $(N_1 + N_2) \cap K$ is the closed unit ball of $N_1 + N_2$, we have from (3.13) that $E(N_1 \cap K) \subseteq E[(N_1 + N_2) \cap K]$. Similarly, $E(N_2 \cap K) \subseteq E[(N_1 + N_2) \cap K]$, and we have equality in (3.16). Finally (3.15) is a consequence of this equality and (3.13).

4. Analytic characterizations of weak* closed L-ideals.

In this section we shall assume that V is a real Banach space, $W = V^*$, K is the closed unit ball of W, and S is the surface of K. We shall assume that K and S are endowed with the weak* topology unless otherwise stated.

For every $v \in V$ we define a weak* continuous function $\tilde{v}$ on K by $\tilde{v}(p) = p(v)$ for $p \in K$, and we recall that $v \mapsto \tilde{v}$ is an isometric isomorphism of V onto $A_0(K)$ (cf. § 1). Thus we can identify V with the subspace $A_0(K)$ of $C(K)$ consisting of all linear functions on K (i.e. affine functions on K vanishing at 0). Accordingly we shall omit the tilde over v and simply write $v(p) = p(v)$, tacitly assuming that the variable p takes values in K.

A function f on K is said to be <u>odd</u> (<u>even</u>) if $f(-p) = -f(p)$ (resp. $f(-p) = f(p)$). We have that $A_0(K)$ consists of the odd functions in $A(K)$. We have the customary decomposition of functions into odd and even components, the <u>odd component</u> of f being

$$(\text{odd } f)(p) = \tfrac{1}{2}(f(p) - f(-p)) \quad \text{for} \quad p \in K \qquad (4.1)$$

The following result is the key lemma of the paper. We note that odd components of upper envelopes were first used by Lazar [La$_1$] to characterize the Banach spaces with L^1 duals.

<u>Lemma 4.1</u>: Suppose that N is a weak* closed subspace of W, and that $v \in V$. Then the function

$$f_v(p) = 2 \text{ odd}(v\chi_{N\cap K} \vee 0)^{\wedge}(p), \qquad p \in K, \qquad (4.2)$$

has the following properties:

(a) $f_v|_{N \cap K} = v$.

(b) $f_v|_{N' \cap S} = 0$.

If N is an L-ideal, then

$$f_v(p) = v(ep) \tag{4.3}$$

where e is the L-projection with $N = eW$. Conversely, if f_v is linear for each $v \in V$, then N is an L-ideal.

<u>Proof</u>. Let $h = v\chi_{N \cap K} \vee 0$. From Lemma 1.2 (a) ,

$$\mathrm{Sub}_K^0 \hat{h} = \overline{\mathrm{co}}(\mathrm{Sub}_K^0 h)$$

We have that

$$\mathrm{Sub}_K^0 h = A \cup B$$

where

$$A = K \times \{0\}$$

$$B = \{(p,\gamma): p \in N \cap K , \quad 0 \leq \gamma \leq v(p)\} .$$

Since A and B are compact convex sets,

$$\overline{\mathrm{co}}(A \cup B) = \mathrm{co}(A \cup B) .$$

Thus an element of $\mathrm{Sub}_K^0 h$ is a convex combination of an element of A and another in B . On the other hand, we claim that each element of A is a convex combination of elements in $(N \cap K) \times \{0\}$ and $(N' \cap K) \times \{0\}$. If $p \notin (N \cap K) \cup (N' \cap K)$, then by Theorem 2.9, $p = q + r$, where $\|p\| = \|q\| + \|r\|$, $0 \neq q \in N$, $0 \neq r \in N'$, and so we have a convex decomposition

$$p = \lambda\left(\frac{\|p\|}{\|q\|}\, q\right) + (1-\lambda)\left(\frac{\|p\|}{\|r\|}\, r\right)$$

where $\lambda = \|q\|\,\|p\|^{-1}$. Clearly every $(p,\gamma) \in B$ is a convex combination of $(p,0)$ and $(p,v(p))$. We conclude that each element of $\mathrm{Sub}_K^0\hat{h}$ has the form

$$(p,\gamma) = \alpha_1(p_1,v(p_1)) + \alpha_2(p_2,0) + \alpha_3(p_3,0) \qquad (4.4)$$

where $p_1 \in N \cap K$ and $v(p_1) \geq 0$, $p_2 \in N \cap K$, $p_3 \in N' \cap K$, and $\alpha_i \geq 0$, $\sum_{i=1}^{3}\alpha_i = 1$.

If p is in K , then $(p,\hat{h}(p))$ is in $\mathrm{Sub}_K^0\hat{h}$, hence selecting p_i and α_i as in (4.4), we have

$$\hat{h}(p) = \alpha_1 v(p_1) \ . \qquad (4.5)$$

Since $\hat{h}$ is a concave function

$$\alpha_1 v(p_1) = \hat{h}(p) \geq \sum_{i=1}^{3} \alpha_i\hat{h}(p_i) \geq \sum_{i=1}^{3} \alpha_i h(p_i) = \alpha_1 v(p_1) + \alpha_2 v(p_2) \ ,$$

and so we can assume $v(p_2) \leq 0$. (If $\alpha_2 = 0$, replacing p_2 by $-p_2$ will not affect (4.4)).

Applying the concavity of $\hat{h}$ once more and observing that $h(-p_1) = 0$ and $h(-p_2) = -v(p_2)$, we get

$$\hat{h}(-p) \geq \sum_{i=1}^{3} \alpha_i\hat{h}(-p_i) \geq \sum_{i=1}^{3} \alpha_i h(-p_i) = -\alpha_2 v(p_2) \ .$$

Combining with (4.5) we obtain the inequality

$$f_v(p) = \hat{h}(p) - \hat{h}(-p) \leq \alpha_1 v(p_1) + \alpha_2 v(p_2) \qquad (4.6)$$

To prove (a) let us suppose that $p \in K \cap N$. Then $\alpha_3 p_3 = p - \alpha_1 p_1 - \alpha_2 p_2 \in N \cap N'$, hence $\alpha_3 p_3 = 0$, and from (4.6), $f_v(p) \leq v(p)$. Similarly $f_v(-p) \leq v(-p)$, and since both f_v and v are odd functions, $f_v(p) = v(p)$.

To prove (b) we consider an arbitrary $p \in N' \cap S$, and we claim that $\alpha_1 p_1 + \alpha_2 p_2 = 0$. If not, let $q_1 = \alpha_1 p_1 + \alpha_2 p_2$, and $q_2 = \alpha_3 p_3$. Noting that $q_2 \neq 0$,

$$p = \|q_1\|(q_1/\|q_1\|) + \|q_2\|(q_2/\|q_2\|) ,$$

and we have $q_1/\|q_1\| \in \mathrm{face}_K(p) \subseteq C(p)$. Since N' is hereditary, $C(p) \subseteq N'$, and $q_1 \in N \cap N'$, a contradiction. Thus from (4.6), $f_v(p) \leq v(0) = 0$. Similarly $f_v(-p) \leq 0$, and since f_v is an odd function, $f_v(p) = 0$.

If N is an L-ideal, $N = eW$ and $N' = (1-e)W$, where e is the corresponding L - projection. It follows that $ep = \alpha_1 p_1 + \alpha_2 p_2$, and from (4.6), $f_v(p) \leq v(ep)$. Since f_v and $v \circ e$ are odd functions we conclude $f_v(p) = v(ep)$.

Suppose that f_v is linear for each $v \in V$. It follows from (b) that $f_v|_{N' \cap K} = 0$ for each $v \in V$. If $p \in N \cap \mathrm{co}(N')$, let $p = \Sigma\, q_i$, $q_i \in N'$. Taking a non-zero multiple of p , we may assume that p and the q_i lie in K. Then for each $v \in V$,

$$v(p) = f_v(p) = \Sigma\, f_v(q_i) = 0 ,$$

hence $p = 0$. From Proposition 3.1, N is an L-ideal.

<u>Corollary 4.2</u>: Suppose that $N = eW$ is a weak* closed L-ideal, with e the corresponding L-projection. Then for each $v \in V$, the function $v \circ e$ weak* Borel on K , and satisfies the barycentric calculus (see § 1).

<u>Proof</u>: This is immediate from Lemma 4.1 and Lemma 1.4.

Lemma 4.3. If C is a weak* closed convex cone in W, then $C \cap S$ and $C' \cap S$ are both weak* Borel subsets of K. If μ is a maximal measure in $P(K)$ then $\mu = \mu|_{C\cap S} + \mu|_{C'\cap S}$.

Proof. It is evident from the definitions of complementary cone and complementary set (see § 2) that $C' \cap S = (C \cap K)^c$. By an elementary theorem in convexity theory,

$$F^c = \{p \in K : \hat{\chi}_F(p) = 0\}$$

for any closed convex subset F of a convex compact set K (see [H, Cor.2.3] or [A_2, Prop. II.6.5]. In these references it was assumed that F was also a face, but this was not used in the proof). It follows that

$$C' \cap S = \{p \in K: \hat{\chi}_{C\cap K}(p) = 0\} \tag{4.7}$$

and since $\hat{\chi}_{C\cap K}$ is upper semi-continuous, $C' \cap S$ is weak* G_δ in K. Letting $C = \{0\}$, it follows that S is weak* G_δ in K, hence for general C, $C \cap S$ is weak* Borel.

From (4.7) we have that

$$\{p \in K: \chi_{C\cap K}(p) = \hat{\chi}_{C\cap K}(p)\} = (C \cap K) \cup (C' \cap S) ,$$

and from Lemma 1.3, this set must carry each maximal $\mu \in P(K)$. Again letting $C = \{0\}$, S carries each maximal measure $\mu \in P(K)$, and the second assertion follows.

If $\mu \in M^+(K)$ and B is Borel in K, we define the restricted measure $\mu|_B$ on K by $\mu|_B(A) = \mu(A \cap B)$ for Borel sets A in K.

Lemma 4.4. Suppose that $N = eW$ is a weak* closed L-ideal, with e the corresponding L-porjection. If μ is a maximal measure in $M^+(K)$ with $r(\mu) = p$ (see § 1), then $\mu = \mu|_{N\cap S} + \mu|_{N'\cap S}$ and

$$r(\mu|_{N\cap S}) = r(\mu|_{N\cap K}) = ep \ ,$$

$$r(\mu|_{N'\cap S}) = (I-e)p \ .$$

Proof: We may assume that $\mu \in P(K)$. From Corollary 4.2, $v\circ e$ satisfies the barycentric calculus. Thus applying (4.3),

$$\begin{aligned} v\circ e(p) &= \int (v\circ e)(q)d\mu(q) \\ &= 2\int \mathrm{odd}(v\chi_{N\cap K} \vee 0)\hat{}\,(q)d\mu(q) \ . \end{aligned}$$

The function $v\chi_{N\cap K} \vee 0 = (v \vee 0)\chi_{N\cap K}$ is upper semi-continuous, hence from Lemma 1.3 and the linearity of v,

$$\begin{aligned} v(ep) &= 2\int_{N\cap K} \mathrm{odd}(v \vee 0)(q)d\mu(q) \\ &= \int_{N\cap K} v(q)d\mu(q) \\ &= \mu|_{N\cap K}(v) \ . \end{aligned}$$

It follows that

$$r(\mu|_{N\cap K}) = ep \ ,$$

hence

$$r(\mu|_{K\setminus N\cap K}) = p - ep \ .$$

Finally from Lemma 4.3,

$$\mu|_{N\cap K} = \mu|_{N\cap S}$$

and

$$\mu|_{K\setminus N\cap K} = \mu|_{N'\cap S} \ .$$

Finally we shall give a characterization of weak* closed L-ideals by means of measures orthogonal to $A_o(K)$. In this connection we recall that a measure μ on K belongs to the annihilator $A_o(K)^{\perp}$ if and only if $r(\mu) = 0$ (see § 1).

Theorem 4.5. A weak* closed subspace N of W is an L-ideal if and only if it is hereditary and satisfies the requirement:

$$\mu \in A_o(K)^{\perp} \text{ and } |\mu| \text{ maximal implies } \mu|_{K\cap N} \in A_o(K)^{\perp} \qquad (4.8)$$

Proof. 1) Assume first that N is an L-ideal, and let μ be a measure in $A_o(K)^{\perp}$ such that $|\mu|$ is maximal. Since $r(\mu^+) = r(\mu^-)$, we have from Lemma 4.4 that $r(\mu^+|_{K\cap N}) = r(\mu^-|_{K\cap N})$, hence $r(\mu|_{K\cap N}) = 0$.

2) Assume next that N is hereditary and satisfies the requirement (4.8). Then there is a well defined map $e : W \to W$ such that

$$e(p) = r(\mu|_{K\cap N}) \qquad (4.9)$$

for every maximal positive measure μ on K with resultant p. e is trivially additive and positively homogeneous. To show that it is linear, we note that the map $p \to -p$ is an affine homeomorphism of K. It induces an isomorphism σ on $M(K)$, and it is quickly verified that if $\mu \in M^+(K)$ is maximal with $r(\mu) = p$, then $\sigma\mu$ is maximal with $r(\sigma u) = -p$. Thus

$$e(-p) = r(\sigma\mu|_{N\cap K}) = r(\sigma(\mu|_{N\cap K})) = r(\mu|_{N\cap K}) = -e(p) .$$

Since $e^2 = e$, e is a projection. We claim that it is an L-projection with range N.

If p is an element of W, we may select a maximal $\mu \in M^+(K)$ with $r(\mu) = p$ and $\|u\| = \|p\|$. To see this, note that if $p = 0$, we may let $\mu = 0$, and if $p \neq 0$, we may let $\mu = \|p\|\mu_o$ is a maximal probability measure on K with $r(\mu_o) = p/\|p\|$. Since we have $\|r(\nu)\| \leq \|\nu\|$ for any $\nu \in M(K)$, It follow that

$$\|p\| = \|ep\| + \|(I-e)p\| . \tag{4.11}$$

The corresponding formula is trivial for $p = 0$. Hence we have proved that e is an L-projection.

Clearly the range of e is contained in N.

To prove $e(W) = N$ we consider an arbitrary non-zero element q in W. Without lack of generality we can assume that $\|q\| = 1$, and we consider a maximal probability measure ν with barycenter q.

By a known result (see e.g. [A_2; Prop. I. 2.3]) the measure ν is weak* limit of discrete probability measures

$$\nu' = \sum_{i=1}^{n} \lambda_i \, \delta_{q_i} ,$$

all with barycenter q.

For any such measure ν' we shall have

$$\|q\| = 1 = \sum_{i=1}^{n} \lambda_i \geq \sum_{i=1}^{n} \lambda_i \|q_i\| \geq \|\sum_{i=1}^{n} \lambda_i \, q_i\| = \|q\| .$$

Hence $q_1, \ldots, q_n$ are without cancellation; and since N is supposed to be hereditary, they must all be in N. It follows that the measure ν must be supported by the weak* closed subspace N. Hence

$$q = r(\nu) = r(\nu|_{K \cap N}) = e(\nu) ,$$

and we are done.

5. Dominated extensions and M-ideals.

As in the preceding section, we will assume that V is a real Banach space, $W = V^*$, and that K is the closed unit ball of W. We define a closed subspace J of V to be an <u>M-ideal</u> if its annihilator J^o in W is an L-ideal. One of the major goals of this section is to give simple geometric characterizations of the M-ideals in V, which do not use W (Theorem 5.8).

Suppose that a is a weak* continuous linear function on a subspace N of W. From the Hahn-Banach Theorem, a has an extension to a weak* continuous linear function $\bar{a}$ on W, i.e., to an element of V. If N is weak* closed, this result can be sharpened. Let N^o be the annihilator of N in V. We may identify N in both the norm and weak* topologies with the Banach dual of V/N^o by letting

$$p(v+N^o) = p(v) \;, \qquad p \in N$$

(this uses the fact that $N^{oo} = N$). Identifying V and N^o with the weak* continuous linear functions on W and N, respectively, we have that for $p \in N$, $(v+N^o)(p) = v(p)$, i.e., the quotient map of V onto V/N^o is just the restriction map $v \mapsto v|_N$. Appealing to the definition of the quotient norm in V/N^o, we have that if a is given as above, and $\varepsilon > 0$, then a is the image of (i.e., has an extension to) an element $\bar{a} \in V$ such that $\|\bar{a}\| \leq \|a\| + \varepsilon$. It is not generally possible to delete the $\varepsilon > 0$. In this section we will show that if N is a weak* closed L-ideal, i.e., $J = N^o$ is an M-ideal, then such isometric extensions exist (see Corollary 5.5). This will be a consequence of a general dominated extension theorem, which has other important applications.

Our first lemma is based on a standard argument. Since we have been unable to find a satisfactory reference, we have included a short proof.

<u>Lemma 5.1.</u> Suppose that a is a weak* continuous linear function defined on a weak* closed linear subspace N of W and that f is a bounded weak* lower semi-continuous convex function on K such that $a(p) < f(p)$ for all $p \in N \cap K$. Then there exists a weak* continuous linear extension $\bar{a}$ of a to all of W such that $\bar{a}(p) < f(p)$ for all $p \in K$.

<u>Proof</u>. Let B be an upper bound for f. The sets $\mathrm{Gr}_N a$ and $\mathrm{Sup}_K^B f$ are disjoint; the former is weak* closed and convex, and the latter is weak* compact and convex. By the Hahn-Banach Theorem there exists a weak* continuous linear function F on $W \times \mathbb{R}$ such that

$$\sup\{F(p,\xi):(p,\xi) \in \mathrm{Gr}_N a\} < \inf\{F(p,\xi):(p,\xi) \in \mathrm{Gr}_K f\} \qquad (5.1)$$

Since $\mathrm{Gr}_N a$ is a linear space $F|_{\mathrm{Gr}_N a} = 0$, and the infimum value at the right hand side of (5.1) is strictly positive.

The function F is of the form

$$F(p,\xi) = v(p) + \alpha\xi \ , \qquad \text{all} \quad (p,\xi) \in W \times \mathbb{R} \ ,$$

where $v \in V$ and $\alpha \in \mathbb{R}$. Since $(0,f(0)) \in \mathrm{Gr}_K f$, we have $0 < F(0,f(0)) = \alpha f(0)$, and since $f(0) > a(0) = 0$, we conclude that $\alpha > 0$. Thus we may define $\bar{a}(p) = -\alpha^{-1} v(p)$ for all $p \in W$.

If $p \in N$, then $(p,a(p)) \in \mathrm{Gr}_N a$, hence

$$0 = F(p,a(p)) = v(p) + \alpha a(p) \ ,$$

giving

$$a(p) = -\alpha^{-1} v(p) = \bar{a}(p) ,$$

i.e., $\bar{a}$ is an extension of a . On the other hand if $p \in K$, then $(p, f(p)) \in Gr_K f$, and so

$$0 < F(p, f(p)) = v(p) + \alpha f(p) = \alpha (f(p) - \bar{a}(p)) .$$

Thus $f(p) > \bar{a}(p)$ for all $p \in K$, and the proof is complete.

<u>Lemma 5.2</u>. Suppose that N is a weak* closed L-ideal in W , g is a weak* lower semi-continuous concave function on K , and that a is a weak* continuous linear function on N such that

$$a(p) < g(p) \qquad \text{all} \quad p \in N \cap K . \tag{5.2}$$

Then a can be extended to a weak* continuous linear function $\bar{a}$ on W satisfying

$$\bar{a}(p) < g(p) \qquad \text{all} \quad p \in K \tag{5.3}$$

if and only if $\breve{g}(0) > 0$.

<u>Proof</u>. 1) To prove necessity, we assume that a has a weak* continuous linear extension $\bar{a}$ which satisfies (5.3). Since g is lower semi-continuous, there is an $\epsilon > 0$ such that $\bar{a} + \epsilon \leq g$. By the definition of lower envelope $\bar{a} + \epsilon \leq \breve{g}$, and so $\breve{g}(0) \geq \epsilon > 0$.

2) To prove sufficiency, we assume $\breve{g}(0) > 0$, and we first observe that we may suppose that $g \leq B$ for some constant $B \geq 0$. Otherwise, choose $b \in A(K)$ with $0 < b(0)$ and $b \leq g$, and replace g by $g_1 = g \wedge (\|a\| + \|b\|)$. Then g_1 is weak* lower semi-continuous and concave, and $b \leq g_1$ implies $0 < b(0) \leq g_1(0)$.

By Lemma 5.1 it suffices to prove that $a(p) < \check{g}(p)$ for all $p \in N \cap K$. To this end we consider a fixed $p \in N \cap K$. By the dual of Lemma 1.2 (c),

$$\check{g}(p) = \inf\{\mu(g) : \mu \in P_p(K)\} \tag{5.4}$$

The mapping $\mu \longmapsto \mu(g)$ is seen to be weak* lower semi-continuous since g is a lower semi-continuous function, and it follows from the weak* compactness of $P_p(K)$ that the infimum value of (5.4) is actually attained at some measure μ. Moreover, we can choose μ to be maximal since g is concave.

By Lemma 4.4 $\mu = \mu|_{N\cap S} + \mu|_{N'\cap S}$ where

$$r(\mu|_{N\cap S}) = ep = p \ , \quad r(\mu|_{N'\cap S}) = (I-e)p = 0 \ .$$

By lower semi-continuity and compactness there exists an $\varepsilon > 0$ such that $a + \varepsilon \leq g|_{N\cap K}$, and we define $\delta = \min(\varepsilon, \check{g}(0))$. It follows that

$$\begin{aligned} \check{g}(p) = \mu(g) &= \mu|_{N\cap S}(g) + \mu|_{N'\cap S}(g) \\ &\geq \mu|_{N\cap S}(a+\varepsilon) + \check{g}(0)\mu(N'\cap S) \geq a(p) + \delta \end{aligned}$$

and we are done.

<u>Lemma 5.3</u>. Suppose that g is a function on K bounded above by a constant $B > 0$, and that b is a weak* continuous linear function on K such that $b(p) < g(p) + \varepsilon$ for all $p \in K$ and some number ε where $0 < \varepsilon \leq 1$. Then

$$(g \wedge b)^{\vee}(0) + \delta\varepsilon > 0 \ , \tag{5.5}$$

where δ is a constant depending on $\check{g}(0)$ and B only.

Specifically, one can take

$$\delta = 1 - \frac{\check{g}(0)}{2(B+1)} \tag{5.6}$$

Proof. Let $C = B + 1$. Then

$$-C \leq b \leq g + \epsilon \leq C \ , \tag{5.7}$$

and the sets $\mathrm{Sup}_K^C \check{g}$ and $\mathrm{Sup}_K^C b$ are compact and convex.

If $a \in A(K)$, then $a \leq g$ if and only if $a \leq \check{g}$, hence it is clear that $(g\wedge b)^{\vee} = (g\wedge b)^{\vee}$. From the dual of Lemma 1.2 (a) and the compactness of the convex hull of two compact convex sets, we have

$$\mathrm{Sup}_K^C (g\wedge b)^{\vee} = \mathrm{co}(\mathrm{Sup}_K^C \check{g} \cup \mathrm{Sup}_K^C b).$$

In particular, if $\eta = (g\wedge b)^{\vee}(0)$, there exists a convex combination

$$(0,\eta) = \alpha(p_1,\eta_1) + (1-\alpha)(p_2,\eta_2) \ ,$$

such that $p_1, p_2 \in K$, $b(p_1) \leq \eta_1$, and $\check{g}(p_2) \leq \eta_2$. Since $0 = b(0) \geq (g\wedge b)^{\vee}(0)$, we have

$$0 \geq \eta \geq \alpha b(p_1) + (1-\alpha)\check{g}(p_2) \ . \tag{5.8}$$

By the convexity of $\check{g}$.

$$\check{g}(0) \leq \alpha \check{g}(p_1) + (1-\alpha)\check{g}(p_2) \ ,$$

hence from (5.8)

$$\check{g}(0) \leq \alpha(\check{g}(p_1) - b(p_1)) \ .$$

By (5.7) this gives $\check{g}(0) \leq 2\alpha C$, and using the definition (5.6) we get $\alpha \geq 1-\delta$. Hence by (5.8) and the inequality

$b(p_2 < g(p_2) + \epsilon$:

$$(g \wedge b)^{\vee}(0) = \eta > \alpha b(p_1) + (1-\alpha)(b(p_2) - \epsilon) \geq -\delta\epsilon \ .$$

Theorem 5.4. Suppose that N is a weak* closed L-ideal in W, g is a weak* lower semi-continuous concave function on K, and that a is a weak* continuous linear function on N such that

$$a(p) \leq g(p) \qquad \text{all} \quad p \in N \cap K \ . \tag{5.9}$$

If $\check{g}(0) > 0$, then a can be extended to a weak* continuous linear function $\bar{a}$ on W such that

$$\bar{a}(p) \leq g(p) \qquad \text{all} \quad p \in K \tag{5.10}$$

Proof. As in the proof of Lemma 5.2, we may assume that g is bounded above by a constant $B > 0$. Since $0 < \check{g}(0) \leq B$, the constant δ defined by (5.6) will satisfy $\frac{1}{2} < \delta < 1$.

From Lemma 5.2 there exists a weak* continuous linear extension b_1 of a to K with $b_1 < g + 1$. Applying Lemma 5.3 with $\epsilon = 1$, we obtain

$$0 < (g \wedge b_1)^{\vee}(0) + \delta = [(g+\delta) \wedge (b_1 + \delta)]^{\vee}(0) \ .$$

Then we can apply Lemma 5.2 with $(g+\delta)\wedge(b_1+\delta)$ in place of g to find a weak* continuous linear extension b_2 of a to K such that $b_2 < g + \delta$ and $b_2 < b_1 + \delta$. Again we may apply Lemma 5.3, this time with $\epsilon = \delta$, and obtain

$$0 < (g \wedge b_2)^{\vee}(0) + \delta^2 = [(g + \delta^2) \wedge (b_2 + \delta^2)]^{\vee}(0) \ .$$

Thus, we may proceed by induction and obtain a sequence $\{b_n\}$ of weak* continuous linear extensions of a such that $b_{n+1} < g + \delta^n$ and $b_{n+1} < b_n + \delta^n$ for $n = 1, 2, \ldots$. Changing signs and using linearity in the latter inequality, we get $\|b_n - b_{n+1}\| < \delta^n$. Hence $\{b_n\}$ is a Cauchy sequence of elements in V, and the function $\bar{a} = \lim_n b_n$ is a weak* continuous linear extension of a. Since $b_{n+1} < g + \delta^n$ on K, we have $\bar{a} \leq g$ on K.

Corollary 5.5. If a is a weak* continuous linear function on a weak* closed L-ideal in W, then a admits a weak* continuous linear extension $\bar{a}$ to W such that $\|\bar{a}\| = \|a\|$.

Proof. Application of Theorem 5.4 with $g = \|a\|$.

Corollary 5.6. If J is an M-ideal of V, then the quotient map of V onto V/J sends the closed unit ball of V onto that of V/J.

Proof. The present corollary is merely a restatement of the preceding one since an element of the closed unit ball of V/J may be regarded as a weak* continuous linear function on $N = J^o$ satisfying $a \leq 1$.

Theorem 5.4 is false if one assumes only that $\check{g}(0) \geq 0$ (see the remarks following Theorem 5.8). Of course the latter is a necessary condition for the existence of $\bar{a}$ (see the proof of Lemma 5.2), and for Lindenstrauss spaces it is both necessary and sufficient (see [L-L, Th.2.2] and statement (V) of the Theorem in [La_1].

If $v \in V$ and $r > 0$, we let $B(v,r)$ and $D(v,r)$ denote the open and closed balls of center v and radius r, respectively.

Lemma 5.7. Suppose that $v_i \in V$, $r_i > 0$, J is a closed subspace of V, $N = J^o$, and $B_i = B(v_i, r_i)$, where $i = 1, \ldots, n$. Then

$$B_1 \cap \ldots \cap B_n \cap J \neq \emptyset \tag{5.11}$$

if and only if

$$[(v_1 + r_1) \wedge \ldots \wedge (v_n + r_n)]^\vee(p) > 0 \quad \text{all} \quad p \in N \cap K \tag{5.12}$$

Proof. If v lies in the intersection, then $\|v - v_i\| < r_i$ implies that $v(p) - v_i(p) < r_i$ for all $p \in K$, i.e.,

$$v(p) < [(v_1 + r_1) \wedge \ldots \wedge (v_n + r_n)](p) \quad \text{all} \quad p \in K .$$

Since $v \in J$, $v|_N = 0$ and (5.12) follows. Conversely, if one has (5.12), then from Lemma 5.1 there exists a weak* continuous linear function v on W extending 0 on N and satisfying $v < g$. It follows that $v \in N^o = J$, and that $v - v_i < r_i$ for $i = 1, \ldots, n$. Changing signs and using linearity, $\|v - v_i\| < r_i$ for $i = 1, \ldots, n$, and $v \in B_1 \cap \ldots \cap B_n \cap J$.

Theorem 5.8. Suppose that J is a closed subspace of a Banach space V and let $\pi : V \to V/J$ be the quotient map. Then the following are equivalent:

(a) J is an M-ideal.

(b) If $B_1, \ldots, B_n$ are open balls with $B_1 \cap \ldots \cap B_n \neq \emptyset$ and $B_i \cap J \neq 0$ for all i, then $B_1 \cap \ldots \cap B_n \cap J \neq \emptyset$.

(c) If $D_1, \ldots, D_n$ are closed balls with $\operatorname{int} D_1 \cap \ldots \cap D_n \neq \emptyset$ and $D_i \cap J \neq 0$ for all i, then $D_1 \cap \ldots \cap D_n \cap J \neq \emptyset$.

(d) If $B_1,\dots,B_n$ are open balls with $B_1\cap\dots\cap B_n \neq \emptyset$, then $\pi(\bigcap_{i=1}^n B_i) = \bigcap_{i=1}^n \pi(B_i)$.

(e) If $D_1,\dots,D_n$ are closed balls with $\operatorname{int} D_1\cap\dots\cap D_n \neq \emptyset$, then $\pi(\bigcap_{i=1}^n D_i) = \bigcap_{i=1}^n \pi(D_i)$.

Proof. We shall attach the subscript n to any of the statements (b) - (d) to indicate the corresponding statement with n fixed. Also we shall write $N = J^o$.

(a)$\Rightarrow$(c). Let $D_i = D(v_i,r_i)$, and $g = (v_1+r_1)\wedge\dots\wedge(v_n+r_n)$. We assume the hypotheses of (c), i.e., $\bigcap_{i=1}^n B(v_i,r_i) \neq \emptyset$, and $D(v_i,r_i) \cap J \neq \emptyset$ for $i = 1,\dots,n$. From the first assumption and Lemma 5.7 with $J = V$, we have that $\check{g}(0) > 0$. From the second, we may select an element $w_i \in D(v_i,r_i) \cap J$. For all $p \in N \cap K$

$$0 = w_i(p) \leq v_i(p) + r_i$$

for $i = 1,\dots,n$, hence $g \geq 0$ on $N \cap K$. Since J is an M-ideal we have from Theorem 5.4 that the function 0 on N can be extended to a weak* continuous linear function v on W such that $v \leq g$. It follows that $v \in D_1\cap\dots\cap D_n \cap J$.

$(c)_n\Rightarrow(b)_n$. Let $B_i = B(v_i,r_i)$, $i = 1,\dots,n$ and say that $v_o \in \bigcap_{i=1}^n B_i$. Letting $\varepsilon > 0$ be such that $B(v_o,\varepsilon) \subseteq \bigcap_{i=1}^n B_i$, we have that $B(v_o,\varepsilon/2) \subseteq \bigcap_{i=1}^n D(v_i,s_i)$ for all $s_i \geq r_i - \varepsilon/2$. On the other hand since in general

$$B(v,r) = \bigcup\{D(v,s) : s < r\}$$

we may for each i select s_i with $r_i - \epsilon/2 \leq s_i < r_i$ and $D(v_i,s_i) \cap J \neq \emptyset$. From $(c)_n$, $\bigcap_{i=1}^{n} D(v_i,s_i) \cap J \neq \emptyset$, hence $\bigcap_{i=1}^{n} B_i \cap J \neq \emptyset$.

(b) $\Longrightarrow$ (a). Let us assume (b) and prove that N is an L-ideal. From Lemma 4.1, it suffices to prove that for each $v \in V$, the function

$$f_v(p) = 2\ \mathrm{odd}(v\chi_{N\cap K} \vee 0)\hat{}\,(p), \qquad p \in K,$$

is linear. We shall do this by showing that f_v is a pointwise limit of a net of linear functions.

Fix $v \in V$ and let $h = v\chi_{N\cap K} \vee 0$. Let $\mathcal{G}_h$ be the net of functions of the form

$$g = (v_1 + r_1)\wedge\ldots\wedge(v_n + r_n) \tag{5.13}$$

with $v_i \in V$, $r_i \in R$, and $h(p) < g(p)$ for all $p \in K$. $\mathcal{G}_h$ is directed downwards by the ordering $\leq$ and thus may be regarded as a decreasing net. From the definition of upper envelope, $\hat{h} = \lim \{g : g \in \mathcal{G}_h\}$. If $g \in \mathcal{G}_h$, $0 < g(p)$ for all $p \in K$, and it follows that $0 < r_i$ and 0 lies in the open ball $B(v_i,r_i)$. Since the open balls $B(v_i,r_i)$ intersect, the same is true for the translated balls

$$B_i = B(v_i - v, r_i) = B(v_i, r) - v .$$

If $g \in \mathcal{G}_h$ is given by (5.13), then $v\chi_{N\cap K} < g$ on K, hence for each i,

$$0 < (v_i - v)(p) + r_i = [v_i - v + r_i]^{\vee}(p)$$

for all $p \in N \cap K$.

From Lemma 5.7, $B_i \cap J \neq \emptyset$. Applying (b), we have an element $w_g \in \bigcap_{i=1}^{n} B_i \cap J$.

Let $v_g = v + w_g$. Then

$$v_g < (v_1 + r_1) \wedge \cdots \wedge (v_n + r_n) = g$$

throughout K, and since $w_g|_N = 0$, and $g > 0$,

$$h = v\chi_{N \cap K} \vee 0 \leq v_g \vee 0 \leq g$$

hence

$$\hat{h} \leq (v_g \vee o)^{\wedge} < g$$

throughout K. It follows that the net $\{(v_g \vee 0)^{\wedge}\}$ with $g \in \mathcal{G}_h$ converges to $\hat{h}$, hence 2 odd $(v_g \vee 0)^{\wedge}$ converges to f_v. Since $N = W$ is a weak* closed L-ideal in W, we have from (4.3) that 2 odd $(v_g \vee 0)^{\wedge} = v_g$, and we are done.

$(c)_n \Rightarrow (e)_n$. If $\pi(v) \in \bigcap_{i=1}^{n} \pi(D_i)$, then $0 \in \bigcap_{i=1}^{n} \pi(D_i - v)$, and for each i, $J \cap (D_i - v) \neq \emptyset$. From $(c)_n$, $J \cap [\bigcap_{i=1}^{n} (D_i - c)] \neq \emptyset$, hence $0 \in \pi[\bigcap_{i=1}^{n} (D_i - v)]$ and $\pi(v) \in \pi[\bigcap_{i=1}^{n} D_i]$. The opposite inclusion is trivial.

$(e)_n \Rightarrow (c)_n$. Since $D_i \cap J \neq \emptyset$, $0 \in \pi(D_i)$. From $(e)_n$, $0 \in \pi(\bigcap_{i=1}^{n} D_i)$, hence $(\bigcap_{i=1}^{n} D_i) \cap J \neq \emptyset$.

$(b)_n \Leftrightarrow (d)_n$ is proved in the same manner as $(c)_n \Leftrightarrow (e)_n$.

Theorem 5.8 can be sharpened. In fact, one has the following

<u>Theorem 5.9:</u> A closed subspace J of a Banach space V is an M-ideal if and only if it satisfies the requirement:

$(b)_3$ If B_1, B_2, B_3 are three open balls with $B_1 \cap B_2 \cap B_3 \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for $i = 1,2,3$, then $B_1 \cap B_2 \cap B_3 \cap J \neq \emptyset$.

The corresponding statement with only two balls is false.

<u>Proof</u>. 1) We adopt the notations of the preceding proof, and we shall give an inductive proof of the sufficiency of $(b)_3$. Specifically, we assume that J satisfies $(b)_{n-1}$ for some $n > 3$, and we shall prove that it also satisfies $(b)_n$.

Let $B_i = B(c_i, r_i)$ for $i = 1,\ldots,n$, and assume that

$$B_1 \cap \ldots \cap B_n \neq \emptyset , \quad B_i \cap J \neq \emptyset \quad \text{for} \quad i = 1,\ldots,n . \tag{5.14}$$

Since the balls B_i are open, there exists an $\varepsilon > 0$ such that the shrinked balls $G_i = B(c_i, r_i - \varepsilon)$ will satisfy

$$G_1 \cap \ldots \cap G_n \neq \emptyset , \quad G_i \cap J \neq \emptyset \quad \text{for} \quad i = 1,\ldots,n . \tag{5.15}$$

Since $n > 3$ it is possible to select four <u>distinct</u> indices from $\{1,\ldots,n\}$, say $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$. Keeping these indices fixed, we define

$$P = \bigcap \{G_i : i \neq i_1, i_2\} , \quad Q = \bigcap \{G_i : i \neq i_3, i_4\} .$$

For $u \in V$ we define

$$\varphi(u) = \max \{d(u,J),\ d(u,P),\ d(u,Q)\} \tag{5.16}$$

and

$$\alpha = \inf_{u \in V} \varphi(u) . \tag{5.17}$$

We claim that

$$\alpha = \inf_{u \in J} (\max \{d(u, P \cap J) ,\ d(u, Q \cap J)\}) . \tag{5.18}$$

To prove (5.18) we let $\eta > 0$ be arbitrary and choose $u_0 \in V$ such that $\varphi(u_0) < \alpha + \eta$. Then the open ball $G = B(u_0,\ \alpha + \eta)$ will satisfy

$$G \cap J \neq \emptyset , \quad G \cap P \neq \emptyset , \quad G \cap Q \neq \emptyset . \tag{5.19}$$

By the definition of P and Q, the two collections:

$$\{G\} \cup \{G_i : i \neq i_1, i_2\}, \quad \{G\} \cup \{G_i : i \neq i_3, i_4\},$$

both consist of $n-1$ balls, and by (5.19) and the induction hypothesis $(b)_{n-1}$, there exist points $v, w \in V$ such that

$$v \in G \cap P \cap J, \qquad w \in G \cap Q \cap J.$$

Defining $u = \frac{1}{2}(v+w) \in J$ and using the fact that $v, w \in G$, we get

$$\|v - w\| \leq \|v - u_0\| + \|u_0 - w\| < 2(\alpha + \eta),$$

and since u is the mid-point of $[v,w]$ this gives

$$\|u - v\| < \alpha + \eta, \qquad \|u - w\| < \alpha + \eta.$$

Hence $d(u, P \cap J) < \alpha + \eta$ and $d(u, Q \cap J) < \alpha + \eta$, and since $\eta > 0$ was arbitrary, this implies

$$\inf_{u \in J} \{d(u, P \cap J), d(u, Q \cap J)\} \leq \alpha,$$

which is the non-trivial half of (5.18).

Next we claim that $\alpha = 0$.

To prove this, we assume for contradiction that $\alpha > 0$. By (5.15) there is a point $d \in V$ such that

$$d \in P \cap Q = G_1 \cap \ldots \cap G_n. \tag{5.20}$$

By the boundedness of the G_i's we can choose a constant $M \geq \alpha$ such that for $a \in V$:

$$\varphi(a) < 2\alpha \quad \text{implies} \quad \|a - d\| \leq M. \tag{5.21}$$

Let δ and λ be positive numbers such that

$$\delta < \frac{\alpha^2}{2M-\alpha} , \qquad \lambda = \frac{2M-\alpha}{2M} , \tag{5.22}$$

and observe that $\delta < \alpha$ and $\lambda < 1$.

By (5.18) there is a point $a \in J$ such that $d(a, P \cap J) < \alpha + \delta$ and $d(a, Q \cap J) < \alpha + \delta$, and we may choose $b \in P \cap J$ and $c \in Q \cap J$ such that

$$\|a-b\| < \alpha + \delta , \qquad \|a-c\| < \alpha + \delta . \tag{5.23}$$

In particular $\varphi(a) < \alpha + \delta < 2\alpha$. Hence $\|a-d\| \leq M$, and by the definition (5.22) of the constant λ, the point $a' = \lambda a + (1-\lambda)d$ will satisfy

$$\|a'-a\| = (1-\lambda)\|d-a\| \leq (1-\lambda)M = \frac{\alpha}{2} < \alpha . \tag{5.24}$$

By (5.22) and (5.23) the points $b' = \lambda b + (1-\lambda)d$ and $c' = \lambda c + (1-\lambda)d$ will satisfy

$$\|a'-b'\| = \lambda\|a-b\| \leq \lambda(\alpha+\delta) < \alpha , \tag{5.25}$$

$$\|a'-c'\| = \lambda\|a-c\| \leq \lambda(\alpha+\delta) < \alpha . \tag{5.26}$$

By the convexity of P and Q , $b' \in P$ and $c' \in Q$. Hence it follows from (5.24), (5.25) and (5.26) that $\varphi(a') < \alpha$. This contradicts the definition (5.17), and we have proved that $\alpha = 0$ as claimed.

Now we may apply (5.18) to find a point $t \in J$ such that $d(t,P) < \varepsilon$ and $d(t,Q) < \varepsilon$. By the definition of P and Q this implies

$$d(t,G_i) < \varepsilon \qquad \text{for } i = 1,\dots,n.$$

Hence $t \in B_i$ for $i = 1,\dots,n$, and so

$$t \in B_1 \cap \ldots \cap B_n \cap J \ ,$$

which completes the induction.

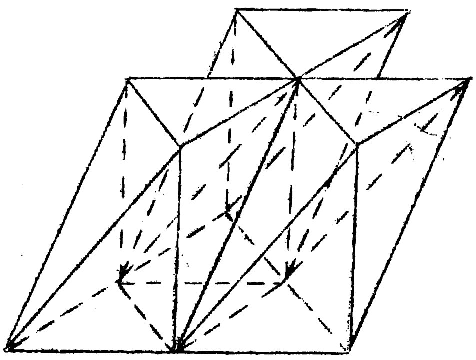
2) We shall prove the insufficiency of $(b)_2$ by a 3-dimensional example. In $\mathbb{R}^3$ we define a norm by the unit ball K consisting of all points (x_1,x_2,x_3) satisfying

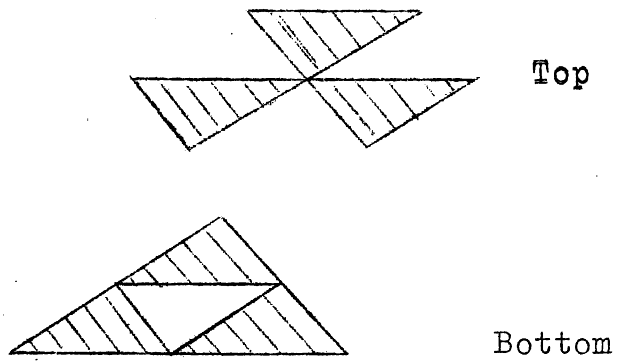
$$-1 \leq \varphi_i(x_1,x_2,x_3) \leq 1 \ , \tag{5.27}$$

where $\varphi_i(x_1,x_2,x_3) = x_i$ for $i = 1,2,3$ and $\varphi_4(x_1,x_2,x_3) = x_1 + x_2 - x_3$. Also we consider the subspace

$$J = \{(x_1,x_2,x_3) : x_3 = 0\} \tag{5.28}$$

The ball K is seen to be an octahedron. We consider three open balls $B_i = B(c_i,1)$ with $c_1 = (1,1,1)$, $c_2 = (1+\beta,1,1)$, and $c_3 = (1,1+\beta,1)$, where $1 < \beta \leq 2$. In the picture these balls are shown for $\beta = 2$.





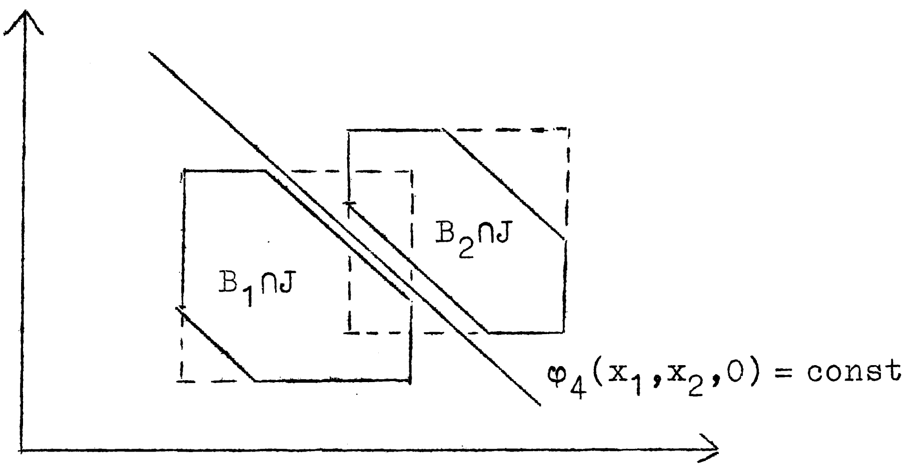
The important part of the diagram is the intersection patterns in the planes $H = \{(x_1,x_2,x_3): x_3 = 2\}$ (top) and $J = \{(x_1,x_2,x_3): x_3 = 0\}$ (bottom). In the former all three sets $\overline{B}_1,\overline{B}_2,\overline{B}_3$ will intersect. In the latter any two of them will intersect, but not all three. Choosing $1 < \alpha < 2$, we get similar intersection patterns for the open balls B_1,B_2,B_3. Hence it follows that J does not satisfy $(b)_3$.

It remains to verify that J satisfies $(b)_2$. By definition a ball is bounded by 4 pairs of parallel planes of the form

$$\varphi_i(x_1,x_2,x_3) = \text{const} \qquad i = 1,2,3,4,$$

and its intersection with J is bounded by the 3 corresponding pairs of lines

$$\varphi_i(x_1,x_2,0) = \text{const} \qquad i = 1,2,4.$$



If B_1 and B_2 are two balls such that $B_1 \cap J$ and $B_2 \cap J$ are disjoint and non-empty, then $B_1 \cap J$ and $B_2 \cap J$ can be separated by some line $\varphi_i(x_1,x_2,0) = \text{const}$, where $i = 1,2,4$. It follows that the balls B_1 and B_2 can be separated by the corresponding plane $\varphi_i(x_1,x_2,x_3) = \text{const}$. Hence $B_1 \cap B_2 = \emptyset$. This proves that J satisfies $(b)_2$, and the proof is complete. (Note that similar octahedra were used in a counterexample of Combes and Perdrizet [P - C]).

Remark 5.10. The statement (c) will no longer characterize M-ideals if one assumes only that $D_1 \cap \ldots \cap D_n \neq \emptyset$. To see this we use an example of Stefánsson, who made an essentially equivalent observation in connection with a problem of Dixmier concerning ideals in a von Neumann algebra [St]. Let V be the ordered Banach space of all self-adjoint operators on a separable Hilbert space H , and J the self-adjoint compact operators. Then J is an M-ideal in V (see § 7). Stefánsson constructed an element k in J for which $-I \leq k \leq I$, and a projection p on H such that $k + p$ is a non-compact projection on H , and $k + p$ is the least upper bound for 0 and k . Since p is a projection, $0 \leq p \leq I$; hence

$$k + p - I \leq k \leq k+p \ , \tag{5.29}$$

$$k \leq k+p \leq k+I \ , \tag{5.30}$$

and since $k + p$ is a projection,

$$0 \leq k+p \leq I \ . \tag{5.31}$$

Letting $D = D_{\frac{1}{2}}(\frac{1}{2})$, we have for any operator $1 \in V$ that $0 \leq 1 \leq I$ if and only if $1 \in D$. From the above inequalities it follows that $k + p$ belongs to the three closed balls

$$D_1 = (k+p-I) + D$$
$$D_2 = k + D$$
$$D_3 = D \ .$$

If v is any element in $D_1 \cap D_2 \cap D_3$, then replacing the central terms in (5.12) and (5.13) by v , we have $0,k \leq v$, hence $k+p \leq v$. Doing the same in (5.11), $v = k+p$. It follows that $D_1 \cap D_2 \cap D_3 \cap J = \emptyset$, even though it is evident that $D_i \cap J \neq \emptyset$ for each i.

If one examines the proof of Theorem 5.7, it becomes apparent that this counter-example shows that Theorem 5.3 is false if one assumes only that $\check{g}(0) \geq 0$. On the other hand, since Theorem 5.4 is true if V is a Lindenstrauss space (see the remark following Corollary 5.6), it follows that this strong intersection property does hold for M-ideals in a Lindenstrauss space.

In the remainder of this section we shall study the properties of M-ideals. The proofs do not use the above characterizations of these subspaces.

<u>Proposition 5.11:</u> Suppose that $\{J_\gamma\}_{\gamma \in \Gamma}$ is a family of M-ideals in V. Then

a) The norm-closure $(\Sigma J_\gamma)^-$ is an M-ideal.

b) If Γ is finite, then $\bigcap J_\gamma$ is also an M-ideal.

<u>Proof:</u> a) is immediate from the relation

$$[(\Sigma J_\gamma)^-]^\circ = \bigcap J_\gamma^\circ \tag{5.32}$$

and (a) of Proposition 3.14. Turning to b), suppose that J_1 and J_2 are M-ideals. $(J_1 \cap J_2)^\circ$ is the weak* closure of $J_1^\circ + J_2^\circ$. On the other hand, from (3.14), it is evident that $(J_1^\circ + J_2^\circ) \cap K$ is weak* compact, hence $J_1^\circ + J_2^\circ$ is weak* closed (see [D.S.,p.429]),

$$(J_1 \cap J_2)^\circ = J_1^\circ + J_2^\circ \tag{5.33}$$

and from (c) of Proposition 3.14, the latter is an L-ideal. An induction argument gives (b).

In contrast to the situation for rings, an intersection of infinitely many M-ideals J_γ need not be an M-ideal (see [Bu; Pe_2]). Since $(\cap J_\gamma)^\circ$ is the weak* closure of ΣJ_γ°, it is necessary and sufficient that this closure be an L-ideal. The uniform closure N of ΣJ_γ° is an L-ideal ((b) of Proposition 3.14) hence it is necessary and sufficient that the weak* closure of the L-ideal N again be an L-ideal.

<u>Proposition 5.12</u>: Suppose that J is an M-ideal in V, and if $J \neq V$, let π be the quotient map of V onto V/J.

(a) If $J \neq \{0\}$, then the M-ideals in J are just the M-ideals in V that are contained in J.

(b) If $J \neq V$, then the π-images of M-ideals are M-ideals, and the inverse images of M-ideals in V/J are the M-ideals in V containing J.

<u>Proof</u>: (a) Let $\iota: J \to V$ be the identity. One may identify the adjoint map $\iota^*: V^* \to J^*$ with the quotient map $\theta: V^* \to V^*/J^\circ$ (see [B_4, p.116]). Taking polars, (a) follows from Proposition 3.15 (d)

(b) The dual map $\pi^*: (V/J)^* \to V^*$ may be identified with the inclusion map $J^\circ \to V^*$ (see [B_4, p.116]). Taking polars, (b) follows from Proposition 3.15 (b).

<u>Proposition 5.13</u>: Suppose that $J_1,\ldots,J_n$ are M-ideals in V with $J_i \neq V$, and let $J = J_1 \cap \ldots \cap J_n$. Then for all $v \in V$

$$\|v + J\| = \max\{\|v + J_i\|: i = 1,\ldots,n\}. \tag{5.34}$$

<u>Proof</u>: We write $N_i = J_i^o$ and $N = J^o$, and we note that $N = \sum_{i=1}^{n} N_i$ (cf. (5.33)). Hence from (3.13) and (3.15):

$$\begin{aligned} E(N \cap K) &= N \cap E(K) = \Big(\sum_{i=1}^{n} N_i\Big) \cap E(K) = \\ &= \bigcup_{i=1}^{n} N_i \cap E(K) = \bigcup_{i=1}^{n} E(N_i \cap K) \end{aligned} \tag{5.35}$$

Given $v \in V$, we may regard it as a weak * continuous linear function on K, as usual. Then $\|v + J\|$ is the uniform norm of $v|_{N \cap K}$, while $\|v + J_i\|$ is the uniform norm of $v|_{N_i \cap K}$ for $i = 1,\ldots,n$. Since maxima and minima of continuous linear functions are attained at extreme points, we may apply (5.35) to obtain (5.34).

<u>Corollary 5.15.</u> If $J_1, \dots, J_n$ are M-ideals in V, then $J_1 + \dots + J_n$ is closed, and thus an M-ideal.

<u>Proof:</u> We may assume $n = 2$. The natural isomorphism $\theta: J_1/J_1 \cap J_2 \mapsto J_1+J_2/J_2$ is an isometry since for every $v \in J$:

$$\phi_{J_1 \cap J_2}(v) = \max\{\phi_{J_1}(v), \phi_{J_2}(v)\} = \phi_{J_2}(v).$$

It follows that J_1+J_2/J_2 is closed in V/J_2, and hence J_1+J_2 is closed in V. Finally, J_1+J_2 is an M-ideal by Proposition 5.11 (a). We define an <u>M-projection</u> e on W to be a linear map of W into itself such that

M_1: e is a projection, i.e., $e^2 = e$

M_2: $\|v\| = \max\{\|ev\|, \|v-ev\|\}$ for all $v \in V$.

Any M-projection e is bounded, and $I-e$ is also an M-projection. If V_1 and V_2 are Banach spaces, and we give the direct sum $V_1 \oplus V_2$ the norm

$$\|(v_1,v_2)\|_\infty = \max\{\|v_1\|,\|v_2\|\}, \tag{5.36}$$

then the operator e on V defined by

$$e(p_1,p_2) = (p_1,0)$$

is an M-projection. After suitable identifications, all M-projections have this form. We say that a subspace of V is an <u>M-summand</u> if it is the range of an M-projection.

It follows from Corollary 5.17 (below), that M-summands are M-ideals. It is not difficult to see that the converse is false. If $V = C(X)$ where X is a compact Hausdorff space, then J is an M-ideal if and only if it is a closed algebraic ideal in the ring $C(X)$ (see § 8). Such ideals are in one-to-one correspondence with closed sets in X, and J is an M-summand if and only if the closed set is also open.

The distinction between M-ideals and M-summands in V has no counterpart for L-ideals and L-summands in the dual space W, since we defined the latter to be the same. However, it is worth noting that we have left open an analogous problem: If N is a closed subspace in a Banach space W with N° an M-summand in W^*, does it follow that N is an L-summand in W? (See Section 10, for a further discussion of this question.)

There is a natural duality between M-projections and L-projections. We have

<u>Proposition 5.16</u>. Suppose that e is a bounded projection on V, and that e^* is the adjoint projection on W. Then

(a) e is an M-projection if and only if e^* is an L-projection.

(b) e is an L-projection if and only if e^* is an M-projection.

<u>Proof</u>. Suppose that e is an M-projection on V, and $p \in W$. Given $\varepsilon > 0$, select $v, w \in V$ with $\|v\|, \|w\| \leq 1$ and

$$e^*p(v) \geq \|ep\| - \varepsilon \ ,$$

$$(p - e^*p)(w) \geq \|p - e^*p\| - \varepsilon \ .$$

Since $\|ev + (I - e)w\| \leq 1$, we have

$$\begin{aligned}\|p\| &\geq p(ev + (I - e)w) \\ &= e^*p(v) + (p - e^*p)(w) \\ &\geq \|e^*p\| + \|p - e^*p\| - 2\varepsilon \ ,\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, it may be deleted. The reverse inequality is trivial, hence e^* is an L-projection.

Next suppose that e is an L-projection, and $p \in V^*$. Given $\epsilon > 0$, select $v \in V$ with $\|v\| \leq 1$ and $p(v) \geq \|p\| - \epsilon$. Then

$$\begin{aligned} \|p\| - \epsilon &\leq e^*p(v) + (I - e^*)p(v) \\ &= e^*p(ev) + (I - e^*)p((I - e^*)v) \\ &\leq \|e^*p\| \, \|ev\| + \|p - e^*p\| \, \|v - ev\| \, . \end{aligned}$$

We have that $\|ev\| + \|v - ev\| = \|v\| \leq 1$, hence

$$\|p\| - \epsilon \leq \max\{\|e^*p\|, \|p - e^*p\|\} \, .$$

Since $\epsilon > 0$ was arbitrary, we may delete it. Since $\|e^*\| = \|e\| \leq 1$, and similarly $\|I - e^*\| \leq 1$, the reverse inequality is evident, and e is an L-projection.

Finally if e is an arbitrary bounded projection, we let e^{**} be the adjoint of e^* in $W^* = V^{**}$. Then regarding V as a subspace of V^{**}, $e = e^{**}|_V$. It follows that if e^* is an L-projection (resp., M-projection), then e^{**} and thus e must be M-projections (resp., L-projections).

Corollary 5.17. An M-summand is an M-ideal.

Proof. For any bounded projection e on V we have

$$(eV)^{\circ} = (I - e^*)W \tag{5.37}$$

The corollary follows from this and (a) of Proposition 5.16.

<u>Corollary 5.18:</u> An M-summand is the range of only one M-projection.

<u>Corollary 5.19:</u> The M-projections commute.

<u>Corollary 5.20:</u> If V is reflexive, every M-ideal J in V is an M-summand.

<u>Proof:</u> J° is an L-ideal in $W = V^{*}$, hence from (5.37) and Proposition 5.15 (b), $J^{\circ\circ}$ is an M-summand in V^{**}. If V is reflexive, we may identify V^{**} with V and $J^{\circ\circ}$ with J.

<u>Corollary 5.21:</u> If J is an M-ideal in V, then it is an M-summand if and only if there is an M-ideal J' with $J \cap J' = \{0\}$ and $J + J' = V$.

<u>Proof:</u> This results from simple applications of Corollary 5.17 and (5.34).

6. Primitive ideals and the structure topologies.

As before, we shall assume that $V^{\cdot}$ is a real Banach space, $W=V$, and K is the closed unit ball of W.

In the preceding section we noted that an M-ideal in V will not generally be an M-summand. We are thus faced with the problem of describing how the M-ideal J and the quotient V/J are "assembled" to form V. As an initial step in this direction we will show that an analogue of the Jacobson structure theory for rings exists for real Banach spaces. To be explicit, we will associate with each real Banach space V a topological space Prim V in such a manner that if J is a M-ideal in V, then Prim V is to within a natural homeomorphism the union of the open set Prim J and the closed set Prim V/J. We will prove in Corollary 6.5 that J is an M-summand in V if and only if the homeomorphic image of Prim J is both open and closed in Prim V, i.e. Prim J is a "direct summand" of the latter topological space.

To every $p\in E(K)$ we associate the largest M-ideal J_p contained in $\ker p$, and we observe that J_p is well-defined by Proposition 5.11(a). If an M-ideal is of the form J_p for some $p\in E(K)$, then it will be said to be <u>primitive</u>. Note that a primitive M-ideal need not be maximal; it is only maximal among the M-ideals contained in some hyperplane of the particular form $\ker p$, where p is an extreme point of K.

We shall also work with the annihilators of the M-ideals in V, i.e. with the weak* closed L-ideals in W, and we note that an M-ideal J^0 is equal to the smallest weak* closed L-ideal N_p containing some point $p\in E(K)$. (N_p is well defined by (3.12)).

Interpreting the elements of V as functions on K in the same way as before, we have for $p\in E(K)$:

$$J_p = \{v\epsilon V: v=0 \quad \text{on} \quad N_p \cap K\} \tag{6.1}$$

By definition J_p is the largest M-ideal of functions in V vanishing at p, and by (6.1) it consists of exactly those functions in V that vanish not only at the point p but on the entire set $N_p \cap K$. By analogy with the theory of rings of continuous functions we may say that primitive M-ideals are "fixed" at points of E(K) (cf. [G.J]). In Section 7 we shall specialize to C^*-algebras V, and we shall see that there a primitive M-ideal is (the self-adjoint part of) the kernel of an irreducible representation.

<u>Lemma 6.1</u>: Suppose that P is a primitive M-ideal in V and that J_1 and J_2 are M-ideals with $J_1 \cap J_2 \subseteq P$. Then either $J_1 \subseteq P$ or $J_2 \subseteq P$ (i.e. P is "prime").

<u>Proof</u>: Let $P=J_p$. Then $p \epsilon P^0 \subseteq (J_1 \cap J_2)^0$, hence from (5.33) and (3.15),

$$p \epsilon (J_1 \cap J_2)^0 \cap E(K) = [J_1{}^0 \cap E(K)] \cup [J_2{}^0 \cap E(K)]$$

It follows that either $p \epsilon J_1{}^0$ or $p \epsilon J_2{}^0$, i.e., either $J_1 \subseteq \ker p$ or $J_2 \subseteq \ker p$. From the definition of a primitive M-ideal $P \supseteq J_1$ or $P \supseteq J_2$.

We denote the set of all primitive ideals in V by Prim V. Since $E(K) \neq \emptyset$, we always have that Prim $V \neq \emptyset$, although it is possible that {0} is the only primitive M-ideal.

If J is an M-ideal in V, the <u>hull of J</u>, h(J), is the set of all primitive M-ideals containing J. We shall say that a subset of Prim J is a <u>hull</u> if it is the hull of some M-ideal. If S is a subset of Prim V, we define the <u>kernel</u> of S, k(S), to be the largest M-ideal contained in $\bigcap\{P: P \epsilon S\}$. The existence of k(S) is assumed by Proposition 5.11(a). It should be noted that in general $\bigcap\{P: P \epsilon S\}$ will not itself be an M-ideal (see the discussion following Proposition 5.11). Hence our notion of kernel is slightly more general than that of ring theory, where the kernel of a collection of ideals is defined to be their intersection. It is readily verified that the operations h and k reverse inclusions, and that $hk(S)\supseteq S$, and $kh(J)\supseteq J$.

Proposition 6.2: The hulls form the closed sets of a topology on Prim V. The closure of a set S in this topology is $hk(S)$.

Proof: Suppose that J_γ is a family of M-ideals. From Proposition 5.11, $(\sum J_\gamma)^-$ is an M-ideal, and it is evident that

$$\bigcap h(J_\gamma) = h\left[(\sum J_\gamma)^-\right] \tag{6.2}$$

If J_1 and J_2 are M-ideals, we have from Proposition 5.11 that $J_1 \cap J_2$ is also an M-ideal. It is evident that $h(J_1)\cup h(J_2)\subseteq h(J_1\cap J_2)$. On the other hand the reverse inclusion is a consequence of Lemma 6.1, i.e.,

$$h(J_1)\cup h(J_2) = h(J_1\cap J_2). \tag{6.3}$$

Note also that $\{0\}$ and V are M-ideals in V with

$$\text{Prim } V = h(\{0\}), \qquad \emptyset = h(V). \tag{6.4}$$

From (6.2), (6.3), (6.4) it follows that the hulls form the closed sets of a topology.

Given a subset S of Prim V, the closure $\bar{S}$ of S is the intersection of the hulls $h(J)$ containing S. Since $hk(S)\supseteq S$, it follows that $hk(S)\supseteq\bar{S}$. Conversely if $h(J)\supseteq S$, then

$$J\subseteq kh(J)\subseteq k(S)$$

and $h(J)\supseteq hk(S)$, hence $\bar{S} = hk(S)$.

We will call the topology on Prim V defined by the hulls the structure topology. We will see in §8 that if V is the self-adjoint part of a C^*-algebra A, then Prim V coincides with the usual Jacobson structure space of A. In particular, we cannot expect this topology to be Hausdorff.

Proposition 6.3: Pulling back the structure topology on Prim V to $E(K)$ by the map $p \mapsto J_p$, one obtains a topology on $E(K)$ whose non-enpty closed sets F are those of the form

$$F = N \cap E(K) = E(N \cap K) \ , \tag{6.5}$$

where N is some weak * closed L-ideal in W with $N \neq \{0\}$.

Proof. A closed set in Prim V is of the form $h(J)$ for some M-ideal J. Letting $N = J^o$ we have $J_p \in h(J)$ if and only if $N_p \subseteq N$, and this is equivalent to $p \in E(K) \cap N$. From this and (3.13), the proposition follows.

The topology introduced in Proposition 6.3, will be called the structure topology on $E(K)$. We note that this topology is analogous to the facial topology in the theory of compact convex sets. In fact the definition is the same, except that weak * closed L-ideals replace closed split faces. (See [A-A] or [A_2].)

Lemma 6.4: Let J be an M-ideal in V with $N = J^o$, let $\pi: V \to V/J$ be the quotient map and let $\iota: J \to V$ be the inclusion map. We denote the unit balls of V, V/J, and J by K, K_1, and K_2, respectively.

(a) If $J \neq V$, then π^* restricts to a structural homeomorphism of $E(K_1)$ onto the structurally cloced set $E(K) \cap N$.

(b) If $J \neq \{0\}$, then ι^* restricts to a structural homeomorphism of the structurally open set $E(K)\ N$ onto $E(K_2)$.

Proof: (a) π^* is an isometry and a weak * homeomorphism of $(V/J)^*$ onto N (see, e.g.,[B_4, p.416]). In particular, it maps $E(K_1)$ onto $E(K \cap N) = E(K) \cap N$, and it carries the weak* closed L-ideals of $(V/J)^*$ onto those of N. From Proposition 3.15(b) and the fact that N itself is weak * closed, the latter are exactly the weak * closed L-ideals of W that are contained in N. From this the first statement follows.

(b) We recall that ι^* is the restriction map going from $W = V^*$ onto J^*. It can be factored $\iota^* = \tau \circ \theta$, where $\theta: W \to W/N$ is the quotient map and τ is an isomorphism. Letting W/N have the quotient weak* topology, τ is an isometry, and a weak* homeomorphism (see [B_4, p.416]). Denoting by N' the complementary L-ideal of N and using Proposition 3.15 (c), we conclude that θ is an isometry of N' onto W/N. Letting $N_1 = N$ and $N_2 = N'$ in (3.15) we get a disjoint union, hence

$$E(K)\setminus N = N' \cap E(K) , \tag{6.5}$$

and $\iota^* = \tau \circ \theta$ is 1-1 from $E(K)\setminus N$ onto $E(K_2)$.

Since τ is an isometry, it maps the L-ideals of W/N onto those of J^*, and it follows from Proposition 3.15 (d) that θ sets up a 1-1 correspondence between tha L-ideals of W containing N and the L-ideals in W/N. From this we conclude that ι^* sets up a 1-1 correspondence between the weak* closed L-ideals of W containing N and the weak* closed L-ideals in J^*.

Finally we observe that if F is a relatively structurally closed subset of $E(K)\setminus N$, then there is a weak* closed L-ideal N_1 in W such that

$$N_1 \supseteq N \quad \text{and} \quad N_1 \cap [E(K)\setminus N] = F .$$

In fact if N_o is a weak* closed L-ideal with $N_o \cap [E(K)\setminus N] = F$, we may let $N_1 = N_o + N$. Then from (3.15),

$$E(K) \cap N_1 = [E(K) \cap N_o] \cup [E(K) \cap N]$$

hence

$$[E(K)\setminus N] \cap N_1 = [E(K)\setminus N] \cap N_o = F$$

From this the second statement of the proposition follows.

Appealing to Proposition 5.12 and the definition of a primitive ideal, we have that for $p \in E(K_1)$, $J_{\pi^*(p)}(J_p)$, and for $p \in E(K)\setminus N$, $J_{\iota^*(p)} = J_p \cap J$. We may conclude from Lemma 6.4:

Proposition 6.5: Suppose that J is an M-ideal of V, and let $\pi:V \to V/J$ be the quotient map.

(a) If $J \neq V$, then $P \to \pi^{-1}(P)$ defines a homeomorphism of Prim (V/J) onto the closed set $h(J)$.

(b) If $J \neq \{0\}$, then $P \to P \cap J$ defines a homeomorphism of the open set Prim $V \setminus h(J)$ onto Prim J.

Corollary 6.6: Suppose that J is an M-ideal in V. Then

(a) $h(J) = \emptyset$ if and only if $J = V$.

(b) $h(J) = \text{Prim } V$ if and only if $J = \{0\}$.

Proof:

(a) If $J \neq V$, then from Proposition 6.5 we have a homeomorphism

$$h(J) \cong \text{Prim } (V/J) \neq \emptyset.$$

The converse is trivial.

(b) If $J = \{0\}$, we similarly have

$$\text{Prim } V - h(J) \cong \text{Prim } J \neq \emptyset.$$

and again the converse is trivial.

Corollary 6.7: Suppose that J is an M-ideal in V. Then J is an M-summand if and only if $h(J)$ is both open and closed in Prim V.

Proof: If e is an M-projection in V with $J = eV$, let $J' = (I-e)V$. Then from (6.2) and (6.3)

$$h(J') \cap h(J) = h(J'+J) = h(V) = \emptyset,$$

$$h(J') \cup h(J) = h(J' \cap J) = h(\{0\}) = \text{Prim } V.$$

Hence $h(J)$ is both open and closed.

Conversely if $h(J)$ is open, let $J' = k[\text{Prim } V \setminus h(J)]$. Then

$$h(J \cap J') = h(J) \cup h(J') = \text{Prim } V,$$

and from Corollary 6.6, $J \cap J' = \{0\}$. On the other hand, $J+J'$ is an M-ideal in V (Corollary 5.14) with

$$h(J+J') = h(J) \cap h(J') = \emptyset,$$

hence $J+J' = V$. Our assertion follows from Proposition 5.19.

As before we denote by N_p the smallest weak*closed L-ideal in W containing a point $p \in K$, and we recall that Z is the weak* closure of E(K) in K (cf. Section 1).

<u>Lemma 6.8</u>: If a net $\{p_\gamma\}$ in E(K) converges to $p \in K \setminus \{0\}$ in the weak topology, then it converges structurally to each $q \in E(K) \setminus N_p$.

<u>Proof</u>: Suppose that $\{p_\gamma\}$ does not converge structurally to q. Then there is a weak*closed L-ideal N such that $q \notin N$ and $p_\gamma \in N$ for arbitrarily large γ. It follows that the weak* limit p must be in N, and so $N_p \subseteq N$. This gives a contradiction since $q \in N_p$.

From Lemma 6.8 we immediately obtain:

<u>Corollary 6.9</u>: Suppose that $p \in Z$ and that q_1, q_2 are any two points of $N_p \cap E(K)$. If U_1, U_2 are structurally open neighbourhoods of q_1 and q_2, respectively, then $U_1 \cap U_2 \neq \emptyset$.

<u>Proposition 6.10</u>: If f is a structurally continuous real valued function on E(K), then it is constant on $E(K) \cap N_p$ for each $p \in Z \setminus \{0\}$, and the extended function $\bar{f}: Z \setminus \{0\} \to \mathbb{R}$ defined by

$$\bar{f}(p) = f(q), \quad \text{for all } q \in E(K) \cap N_p, \tag{6.6}$$

is weak* continuous.

<u>Proof</u>: We only have to prove continuity. For a closed subset F of $\mathbb{R}$ there exists a weak* closed L-ideal N in W such that $f^{-1}(F) = E(K) \cap N$. For $p \in K \setminus \{0\}$ one has $p \in N$ if and only if $E(K) \cap N_p \subseteq f^{-1}(F)$. Hence

$$(\bar{f})^{-1}(F) = (Z \setminus \{0\}) \cap N,$$

and weak continuity is proved.

7. The centralizer

As in the preceding section, we shall assume that V is a Banach space, $W = V^*$, and that K is the closed unit ball of W. Regarded as a function on K, any element v of V must assume its maximum and minimum values on $E(K)$. Letting Z be the weak* closure of $E(K)$, we have that $v \mapsto v|_Z$ is a linear isometry of V onto a closed linear subspace of $C(Z)$. This representation of V by functions on Z has proved particularly valuable in the theory of Lindenstrauss spaces (see [E_1]).

Since there is no positive function in V other that $v = 0$, this space can not be pointwise ordered. Nonetheless, there is a natural partial ordering available, which is in some respects dual to the domination ordering $\dashv$ for W (see § 2).

Let us first consider $\mathbb{R}$ as a real Banach space. Clearly, two elements λ, μ of $\mathbb{R}$ are without cancellation if and only if $\operatorname{sign} \lambda = \operatorname{sign} \mu$, and $\lambda \dashv \mu$ if and only if $0 \leq \lambda \leq \mu$ or $\mu \leq \lambda \leq 0$. Passing to a general function space, one may define "M-cancellation" and "M-ordering" by requiring these relations to hold pointwise.

More specifically, we shall say that two elements u, v of V are without <u>M-cancellation</u>, and we shall write $u|_M v$ if for all $p \in Z$:

$$\operatorname{sign} u(p) = \operatorname{sign} v(p) . \tag{7.1}$$

Similarly we shall say that u is <u>M-dominated</u> by v, and we shall write $u \dashv_M v$ if for all $p \in Z$:

$$0 \leq u(p) \leq v(p) \quad \text{or} \quad v(p) \leq u(p) \leq 0 . \tag{7.2}$$

<u>Remark</u>. Clearly it would suffice to require (7.1) and (7.2) on $E(K)$, but it is convenient to work in the compact space Z.

Note, however, that one can not replace Z by all of K, nor with the entire surface $S = \{p \in K: \|p\| = 1\}$. In fact if (7.1) or (7.2) are valid for all $p \in K$ (or all $p \in S$), then $\ker u \supseteq \ker v$ and u and v must differ only by some scalar factor.

Lemma 7.1. The relation $u \prec_M v$ is a partial ordering on V, and the formulas (2.3)-(2.6) are all valid with $|_M$ in the place of $|$ and $\prec_M$ in the place of $\prec$.

The proofs are trivial.

Proposition 7.2. If $v, w \in V$, then the following are equivalent:

(a) $v \prec_M w$

(b) $v \leq (w \vee 0)^\wedge$

(c) If D is any closed ball containing 0 and w, then D contains v.

Proof. (a) $\Longrightarrow$ (b). Suppose that $v \prec_M w$. Then for all $p \in Z$,

$$v(p) \leq w(p) \vee 0$$

Given $p \in K$, let $\mu \in P(K)$ be maximal with $r(\mu) = p$ and

$$\mu(w \vee 0) = (w \vee 0)^\wedge(p) .$$

Then

$$v(p) = \mu(v) \leq \mu(w \vee 0) = (w \vee 0)^\wedge(p) .$$

(b) $\Longrightarrow$ (a). If $v \leq (w \vee 0)^\wedge$, then for all $p \in E(K)$

$$v(p) \leq (w \vee 0)^\wedge(p) = (w \vee 0)(p) .$$

If $w(p) \geq 0$, it follows that $v(p) \leq w(p)$ and

$$v(-p) \leq w(-p) \vee 0 = 0 \ ,$$

hence $0 \leq v(p) \leq w(p)$. If $w(p) \leq 0$, substition of $-p$ for p , gives $w(p) \leq v(p) \leq 0$. These alternative inequalities extent by continuity to all $p \in Z$; hence from (7.2), $v \prec_M w$.

(b) $\Longleftrightarrow$ (c) . Let us fix $w \in V$. Given $a \in V$ and $r \geq 0$, we have that for any $b \in V$, b lies in $D(a,r)$ if and only if $(b-a)(p) \leq r$ for all $p \in K$, i.e. $b \leq a+r$. Thus 0 and w both lie in $D(a,r)$ if and only if $0 \vee w \leq a+r$. It follows that (c) is equivalent to the assertion that if $a \in V$ and $r \geq 0$ are such that $0 \vee w \leq a+r$, then $v \leq a+r$. Since any continuous affine majorant of $0 \vee w$ has the form $a+r$, this is equivalent to (b).

<u>Corollary 7.3</u>. Suppose that $\mathbb{R} Z = W$ (this is the case if $E(K)$ is dense in the surface S of K and in particular if K is strictly convex). Then for any two points $v_1, v_2 \in V$, the intersection of all closed balls containing v_1 and v_2 is the line segment joining those points.

<u>Proof</u>. Using a translation, we may assume that $v_1 = 0$. From Proposition 7.2 it suffices to show that if $u \prec_M v_2$, then u is a multiple of v_2 . But this is evident since the relation (7.2) for all $p \in Z$ implies the same for all $p \in W$ in the present situation.

<u>Corollary 7.4</u>. If $v \prec_M w$, then $\|v\| \leq \|w\|$.

<u>Proof</u>. The closed ball $D(0,w)$ contains 0 and w .

By analogy with § 3, we say that two linear operators S, T on V are <u>without M-cancellation</u> if $Sp \mid_M Tp$ for each $p \in V$. We write $S \mid_M T$ if S and T are without M-cancellation, and $S \prec_M T$ if $S \mid_M (T-S)$, or equivalently, $Sp \prec_M Tp$ for all $p \in V$. It is quickly verified that $\prec_M$ is a partial ordering on the linear operators, satisfying analogues of (2.3)-(2.6). In addition

$$\alpha I \mid_M \beta I \quad \text{for all} \quad \alpha, \beta \geq 0 .$$

We will next characterize the linear operators T on V such that $T \prec_M \alpha I$ for some $\alpha \geq 0$. We note that due to (7.2), such operators are bounded.

We let $\mathcal{B}(V)$ be the Banach algebra of bounded linear operators on V with the uniform norm. We recall that for any $T \in \mathcal{B}(V)$ the adjoint operator $T^*: W \mapsto W$ is defined by

$$(T^*p)(v) = p(Tv) , \tag{7.3}$$

where $v \in V$ and $p \in W$. If we identify V with the weak* dual of W, we may rewrite (7.3) as follows:

$$v(T^*p) = (Tv)(p) . \tag{7.4}$$

We shall need the following elementary fact, which we state as a proposition for later references.

<u>Proposition 7.3</u>. The map $T \mapsto T^*$ is an isometric anti-isomorphism of $\mathcal{B}(V)$ onto the space of weak* continuous linear operators on W.

<u>Proof</u>. The proof is a straightforward verification, except perhaps for the surjectivity. If $S: W \mapsto W$ is a weak* continuous

linear operator, then $S(K)$ is weak* compact, and so there is a $\rho \in \mathbb{R}^+$ such that $S(K) \subset \rho K$. Then the operator S_* defined on V by

$$(S_* v)(p) = v(Sp) \tag{7.5}$$

for $v \in V$ and $p \in W$, will be bounded with $\|S_*\| \leq \rho$, and it follows from (7.4) and (7.5) that $(S_*)^* = S$.

An operator $T \in \mathcal{B}(V)$ will be said to be a <u>multiplier</u> if there exists a real valued function λ on $E(K)$ such that

$$(Tv)(p) = \lambda(p) \cdot v(p) , \tag{7.6}$$

for all $v \in V$ and $p \in E(K)$. (Note that the multiplication takes place on the set of extreme points of K only.) By virtue of (7.4) one may rewrite this definition in the form:

$$T^*p = \lambda(p)p , \quad \text{for all } p \in E(K) . \tag{7.7}$$

<u>Thus an operator $T \in \mathcal{B}(V)$ is a multiplier if and only if every $p \in E(V)$ is an eigenvector for T^*</u>. If T is a multiplier we shall use the symbol $\tilde{T}$ to denote the real-valued function λ on $E(K)$ associated with T.

<u>Lemma 7.4</u>. Suppose that T is a linear operator on V, and $\alpha \geq 0$. Then the following are equivalent:

(a) $T \precsim_M \alpha I$

(b) T is bounded, and $T^* \precsim \alpha I$

(c) T is a multiplier with $0 \leq \tilde{T} \leq \alpha$.

<u>Proof</u>. (a) $\Longleftrightarrow$ (c). We have that $T \precsim_M \alpha I$ if and only if for each $v \in V$, $Tv \precsim_M \alpha v$, i.e., for all $p \in E(K)$

$$(T^*p)(v) = (Tv)(p) \precsim (\alpha v)(p) = (\alpha p)(v) . \tag{7.8}$$

If T is a multiplier with $0 \leq \tilde{T} \leq \alpha$, then for $p \in E(K)$,

$$(T^*p)(v) = \tilde{T}(p)\, p(v) \precsim \alpha p(v) ,$$

and so $T \precsim_M \alpha I$.

Conversely, from (7.8) we have $\ker T^*p \supseteq \ker \alpha p$, hence there is a scalar λ with $T^*p = \lambda p$. If we select $v \in V$ with $T^*p(v) \geq 0$, it is apparent from (7.8) that $0 \leq \lambda \leq \alpha$. Thus T is a multiplier with $0 \leq \tilde{T}\,(p) \leq \alpha$.

(b) $\Longrightarrow$ (c) . If T is bounded with $T^* \precsim \alpha I$, then for each $p \in K$, $T^*p \precsim \alpha p$. In particular if $p \in E(K)$, then $C(p) = \mathbb{R}^+ p$ and there is a scalar $0 \leq \lambda \leq \alpha$ with $T^*(p) = \lambda p$, i.e., we have (c) .

(c) $\Longrightarrow$ (b) . Assuming (c), it suffices to prove that for each $p \in K$,

$$\|T^*p\| + \|\alpha p - T^*p\| \leq \alpha\|p\| .$$

(The reverse inequality is trivial.)

We first assume that $p \in \operatorname{co} E(K)$ and $\|p\| = 1$. Then p can be written as a proper convex combination $p = \Sigma_{i=1}^n \beta_i p_i$, where $p_1, \ldots, p_n \in E(K)$. In particular, $p_i \in \operatorname{face}_K\{p\}$. Given positive scalars α_i , we have $\alpha_i p_i \in C(p)$, hence from Lemma 2.3,

$$\|\Sigma \alpha_i p_i\| = \Sigma \alpha_i \|p_i\| = \Sigma \alpha_i .$$

Since $0 \leq \tilde{T}(p_i)$ and $0 \leq \alpha - \tilde{T}(p_i)$, we conclude that

$$\begin{aligned}\|T^*p\| &+ \|\alpha p - T^*p\| \\ &= \|\Sigma \beta_i \tilde{T}(p_i) p_i\| + \|\Sigma \beta_i (\alpha - \tilde{T}(p_i)) p_i\| \\ &= \alpha \Sigma \beta_i = \alpha = \alpha\|p\| .\end{aligned}$$

We assume next that $p \in K$ and $\|p\| = 1$. By the Krein-Milman Theorem there is a net $\{p_\gamma\}$ from $\operatorname{co} E(K)$ which converges to p in the weak* topology. It is well known (and easily verified) that the norm function on K is weak* lower semi-continuous. Hence

$$1 = \|p\| \leq \liminf_\gamma \|p_\gamma\| \leq \limsup_\gamma \|p_\gamma\| \leq 1 ,$$

and so $\|p_\gamma\| \to 1$. Defining $q_\gamma = \|p_\gamma\|^{-1} p_\gamma$, we obtain a net $\{q_\gamma\}$ which converges to p in the weak* topology, and such that $\|q_\gamma\| = 1$ and $q_\gamma \in \operatorname{co} E(K)$ for all γ. From the preceding results

$$\begin{aligned} \|T^*p\| + \|p - T^*p\| &\leq \liminf_\gamma (\|T^*q_\gamma\| + \|\alpha q_\gamma - T^*q_\gamma\|) \\ &= \liminf_\gamma \alpha \|q_\gamma\| \\ &= \alpha \|p\| . \end{aligned}$$

A linear operator $T: V \to V$ is said to be <u>M-bounded</u> if there exists an $\alpha > 0$ such that for any given $v \in V$, the point Tv is contained in every closed (or open) ball containing λv and $-\lambda v$.

This definition is seen to reduce to the customary definition boundedness if we specialize to balls centered at the origin. Hence, <u>every M-bounded linear operator is bounded</u>. Note, however, that the opposite is false. If the hypothesis of Corollary 7.3 is satisfied, then the only M-bounded operators on V are the scalars.

The M-bounded linear operators on V form a linear space, which we shall call the <u>centralizer</u> of V, and denote by $\mathcal{Z}(V)$.

<u>Lemma 7.5</u>. $\mathcal{Z}(V)$ is the linear span of the cone $\mathcal{Z}(V)^+$ of all $T \in \mathcal{B}(V)$ for which $T \prec_M \alpha I$ for some scalar $\alpha \geq 0$. Moreover, $\mathcal{Z}(V)$ is an Archimedean order unit space with the ordering defined by $\mathcal{Z}(V)^+$ and the unit I.

Proof. It follows from Proposition 7.2 that $\mathcal{Z}(V)^+ \subseteq \mathcal{Z}(V)$.

Assume next that $T \in \mathcal{Z}(V)$, say that for all $v \in V$:

$$Tv \in \bigcap\{D:\ -\lambda v, \lambda v \in D\}\,. \tag{7.9}$$

Modifying T by a translation and a scalar multiplication, we arrive at an operator $S = (2\lambda)^{-1}(T + \lambda I)$ satisfying:

$$Sv \in \bigcap\{D:\ o, v \in D\}\,, \tag{7.10}$$

for all $v \in V$.

By Proposition 7.2, $S \precsim_M I$. Hence $T + \lambda I = 2\lambda S \precsim_M 2\lambda I$, and the decomposition $T = (T + \alpha I) - \alpha I$ proves $T \in \mathcal{Z}(V)^+ - \mathcal{Z}(V)^+$.

An argument similar to the proof of Lemma 3.5, but this time based on Lemma 7.1, shows that the ordering $\leq$ on $\mathcal{Z}(V)$ defined by the cone $\mathcal{Z}(V)^+$, will coincide with $\precsim_M$ on $\mathcal{Z}(V)^+$. From this it easily follows that I is an order unit on $\mathcal{Z}(V)$ (see the first part of the proof of Lemma 4.3). To prove Archimedicity we assume that $S, T \in \mathcal{Z}(V)^+$ are such that $S - T \leq \epsilon I$ for all $\epsilon > 0$. This means that $0 \leq S \leq T + \epsilon I$. Hence $S \precsim_M T + \epsilon I$, and so for any $v \in V$, $Sv \precsim_M Tv + \epsilon v$. For an arbitrary $p \in Z$ we get by definition

$$0 \leq (Sv)(p) \leq (Tv)(p) + \epsilon v(p)\,,$$

or

$$(Tv)(p) + \epsilon v(p) \leq (Sv)(p) \leq 0\,.$$

Since $\epsilon > 0$ is arbitrary, the same relations must hold with $\epsilon = 0$. Hence $Sv \precsim_M Tv$, and since $v \in V$ is arbitrary $S \precsim_M T$. This means $0 \leq S \leq T$, and so $S - T \leq 0$.

Theorem 7.6. For $T \in \mathcal{B}(V)$ the following statements are equivalent:

(a) T is M-bounded

(b) T^* is in the Cunningham algebra $\mathcal{C}(W)$

(c) T is a multiplier

In addition, $\mathcal{Z}(V)$ is a closed commutative subalgebra of $\mathcal{B}(V)$, and it is a lattice in the ordering defined by the cone $\mathcal{Z}(V)^+$. Moreover, $T \mapsto T^*$ is an isometric algebra- and lattice-isomorphism of $\mathcal{Z}(V)$ onto the space of all weak* continuous operators in $\mathcal{C}(W)$, and $T \mapsto \tilde{T}$ is an isometric algebra- and lattice- isomorphism of $\mathcal{Z}(V)$ into the space of all bounded real valued functions on $E(K)$.

Proof. For brevity we denote by $\mathcal{H}$ the space of weak* continuous operators in $\mathcal{C}(W)$ and by $\mathcal{F}$ the space of all real valued functions on $E(K)$ which are of the form $\tilde{T}$ for some multiplier T on V. The space $\mathcal{Z}(V)$ is an Archimedean order unit space. Also the space $\mathcal{H}$ is an Archimedean order unit space (see § 3), and clearly $\mathcal{F}$ is the same.

It follows from Lemma 7.4 that the maps $T \mapsto T^*$ and $T \mapsto \tilde{T}$ are linear, and they map the positive part of the unit ball of $\mathcal{Z}(V)$ 1 - 1 onto the corresponding sets in $\mathcal{H}$ and $\mathcal{F}$. Hence the two maps are isometric order isomorphisms. (The ordering and the norm of an Archimedean order unit space are both determined by the positive part of the unit ball. See e.g. [A_2, Ch II, § 1].) Note that the maps $T \mapsto T^*$ and $T \mapsto \tilde{T}$ are isometries with respect to the order unit norm of $\mathcal{Z}(V)$. However, since the order unit norm and the operator norm on $\mathcal{C}(W)$ are known to coincide (see § 3), and the operator norms satisfy the equation $\|T^*\| = \|T\|$, the two norms on $\mathcal{Z}(V)$ must also coincide.

The space $\mathcal{H}$ is seen to be a norm closed subalgebra of $\mathcal{C}(W)$. Hence $\mathcal{Z}(V)$ must be a complete, and hence closed, subspace of $\mathcal{B}(V)$.

It is easily verified that the product of two multipliers S and T is a multiplier, and that $(ST)^{\sim} = \tilde{S}\tilde{T}$. Hence $\mathcal{Z}(V)$ is a subalgebra of $\mathcal{B}(V)$, and the map $T \mapsto \tilde{T}$ is an algebra isomorphism of $\mathcal{Z}(V)$ onto $\tilde{\mathcal{F}}$. In particular $\mathcal{Z}(V)$ is a commutative subalgebra of $\mathcal{B}(V)$. From this we also conclude that for any two elements S, T of $\mathcal{Z}(V)$:

$$(TS)^* = (ST)^* = T^* S^* .$$

Thus $T \mapsto T^*$ is an algebra isomorphism of $\mathcal{Z}(V)$ onto $\mathcal{H}$.

Finally we observe that since $\tilde{\mathcal{F}}$ is a norm closed algebra of bounded real valued functions on the set $E(K)$, it is also a vector lattice under pointwise lattice operations, and the absolute value is related to the multiplicative structure by the formula $|\tilde{T}| = \sqrt{\tilde{T}^2}$, where the square root can be calculated by a binomial series in the usual way. Similarly we conclude that the norm closed subalgebra $\mathcal{H}$ of $\mathcal{C}(W)$ is closed under the lattice operations of $\mathcal{C}(W)$, and that $|T^*| = \sqrt{(T^*)^2}$, where the square root can be calculated by a binomial series. From this it follows that $\mathcal{Z}(V)$ is a vector lattice in the given ordering and that $T \mapsto T^*$ and $T \mapsto \tilde{T}$ preserve lattice operations.

Remark It follows from standard representation theorems (e.g. from Lemma 4.7), that the Gelfand transform is an isometric order- and algebra- isomorphism of $\mathcal{Z}(V)$ onto $C(X)$, where X is the spectrum (maximal ideal space) of $\mathcal{Z}(V)$.

We now proceed to determine the range of the mapping $T \mapsto \tilde{T}$ studied in Theorem 7.6. The answer to this problem will be a general form of the Dauns-Hofmann Theorem.

Theorem 7.7. Suppose that V is a real Banach space. Then $T \mapsto \tilde{T}$ is an isometric algebraic isomorphism of $\mathcal{Z}(V)$ onto $C_s^b(E(K))$, the bounded structurally continuous functions on $E(K)$.

Proof. From Theorem 7.6., it suffices to show that $T \mapsto \tilde{T}$ maps $\mathcal{Z}(V)$ onto $C_s^b(E(K))$.

(1) Suppose first that $T \in \mathcal{Z}(V)$. Since $(\alpha T + \beta I)^{\sim} = \alpha\tilde{T} + \beta$ for all pairs α, β of real numbers, it suffices to show that

$$C = \{p \in E(K) : \tilde{T} \geq 1\}$$

is structurally closed. We have

$$C = \{p \in E(K) : (\tilde{T} \vee 0) \wedge 1(p) = 1\}$$

Letting $S = (T \vee 0) \wedge I$, S is an element of $\mathcal{Z}(V)$ for which $0 \leq S \leq I$, and due to Theorem 7.6,

$$C = \{p \in E(K) : \tilde{S}(p) = 1\} .$$

Letting

$$N = \{p \in V^* : S^*(p) = p\} ,$$

we have that $C = N \cap E(K)$. Since N is weak* closed, it suffices to show that N is an L-ideal.

We have that S^n, $n = 1,2,\ldots$ is a decreasing sequence in $\mathcal{C}(W)$ with $0 \leq S^n \leq I$. From Lemma 3.10, S^n converges strongly to an operator $e \in \mathcal{C}(W)$. It is apparent from the inequality

$$\|U\ V\ p - U_oV_o\| \leq \|U\|\ \|Vp - V_op\| + \|(U - U_o)(V_op)\| \tag{7.11}$$

that multiplication is jointly continuous in the strong topology on $\mathcal{C}(W)$. It follows that $e^2 = e$, i.e. e is an L-projection.

We claim that $N = eW$. If $p \in N$, then $p = S^n p$ converges in norm to ep, hence $p \in eW$. Conversely, if $p \in eW$, then since $e \dashv S \dashv I$, we have $p = ep \dashv Sp \dashv p$, i.e., $Sp = p$ and $p \in N$.

(2) Suppose that φ is a structurally continuous real function on $E(K)$. We shall construct an operator $T \in \mathcal{Z}(V)$ with $\tilde{T} = \varphi$. It will suffice to construct an operator $S \in \mathcal{C}(W)$ such that

$$Sp = \varphi(p)p \qquad \text{for all } \ p \in E(K) \tag{7.12}$$

and which is weak* continuous, since then the $T \in \mathcal{B}(V)$ with $T^* = S$ will be the desired operator. It is clearly no restriction to assume that $0 \leq \varphi \leq 1$.

If F is structurally closed in $E(K)$, let $N(F)$ be the weak*-closed L-ideal with $F = N(F) \cap E(K)$, and $e(F)$ be the L-projection with $N(F) = e(F)W$. Due to the Krein-Milman Theorem and (3.13), $N(F)$ and thus $e(F)$, are uniquely determined. If F_1 and F_2 are structurally closed,

$$N(F_1) + N(F_2) = [e(F_1) \vee e(F_2)]W$$

is a weak* closed L-ideal with

$$F_1 \cup F_2 = [N(F_1) + N(F_2)] \cap E(K)$$

(see (3.15)). Thus

$$e(F_1 \cup F_2) = e(F_1) \vee e(F_2) \, . \tag{7.13}$$

We also have for $p \in E(K)$,

$$e(F)(p) = \chi_F(p)p$$

since if $p \notin F \subseteq e(F)W$, then $p \in [I - e(F)]W \cap E(K)$ (see (3.15)).

We next approximate φ be step functions. For $n = 1,2,\ldots$ and $j = 1,\ldots,2^n$, the set

$$F_{n,j} = \{p \in E(K) : \varphi(p) \geq j\,2^{-n}\} ,$$

is structurally closed. It is evident that

$$F_{n,j} = F_{n+1,2j} \supseteq F_{n+1,2j+1} \supseteq F_{n,j+1} \tag{7.14}$$

We shall let

$$f_n = 2^{-n} \Sigma_{j=1}^{2^n} \chi_{F_{n,j}}$$

and

$$S_n = 2^{-n} \Sigma_{j=1}^{2^n} e(F_{n,j}) .$$

Then we have that for $p \in E(K)$,

$$|f_n(p) - \varphi(p)| \leq 2^{-n} , \tag{7.15}$$

$$S_n(p) = f_n(p)p , \tag{7.16}$$

and

$$\|S_{n+1} - S_n\| \leq 2^{-n} \tag{7.17}$$

In order to prove the last inequality, fix n and note that

$$2^{-n}e(F_{n,j}) - 2^{-(n+1)}[e(F_{n+1,2j}) + e(F_{n+1,2j+1})]$$

$$= 2^{-(n+1)}[e(F_{n,j}) - e(F_{n+1,2j+1})]$$

$$= 2^{-(n+1)}e_{n,j} ,$$

where the $e_{n,j}$, $j = 1,\ldots,2^n$ are <u>disjoint</u> L-projections. We have that

$$S_{n+1} - S_n = 2^{-(n+1)}e(F_{n+1,1}) + 2^{-(n+1)}\Sigma_j e_{n,j} ,$$

hence using the fact that non-zero L-projections have norm 1,

$$\|S_{n+1} - S_n\| \leq 2^{-(n+1)}\|e(F_{n+1,1})\| + 2^{-(n+1)}\|\Sigma_j e_{n,j}\|$$
$$\leq 2^{-n} .$$

It follows from (7.15) - (7.17) that the operators S_n converge uniformly to an operator $S \in \mathcal{B}(W)$ satisfying (7.12).

We next show that S is weak* continuous, i.e. that for each $v \in V$, it suffices to prove that $v(Sp)$ is weak* continuous on K. For each n we have that

$$v(S_n p) = \frac{1}{2^n} \Sigma_j \; v(e(F_{n,j})p) ,$$

and from Corollary 4.2, $v(S_n p) \in \mathcal{A}(K)$. Since the functions $v(S_n p)$ converge uniformly to $v \circ S$, $v \circ S \in \mathcal{A}(K)$ (see § 1). From Lemma 1.4 it thus suffices to prove that $v \circ S$ is continuous on $Z = E(K)^-$. It follows from Proposition 6.10 that φ has a weak* continuous, bounded extension $\bar{\varphi}$ on $Z \setminus \{0\}$. If $0 \in Z$, we define $\bar{\varphi}(0) = 0$. Although this may introduce a discontinuity, the function $\bar{\varphi}(p)v(p)$ is continuous on Z, and it will suffice to prove that

$$v(Sp) = \bar{\varphi}(p)v(p) \tag{7.18}$$

for $p \in Z$.

We begin by verifying (7.18) on a measure-theoretically well-behaved set containing $E(K)$ (this is unnecessary if K is metrizable, i.e. V is separable, since one may then use $E(K)$ itself). If $N = eW$ is a weak* closed L-ideal, the set

$$D(e) = \{p \in K : v(ep) = \chi_{N \cap K}(p)v(p)\}$$

contains $E(K)$. On the other hand since

$$v(ep) = 2\mathrm{odd}(v\chi_{N\cap K} \vee 0)^\wedge(p)$$

and

$$\chi_{N\cap K}(p)v(p) = 2\mathrm{odd}(v\chi_{N\cap K} \vee 0)(p) ,$$

we have that the set $D(e)$ is Borel and

$$D(e) \supseteq \{p : (v\chi_{N\cap K} \vee 0)^\wedge(p) = (v\chi_{N\cap K} \vee 0)(p)\}$$
$$\cap \{p : (v\chi_{N\cap K} \vee 0)^\wedge(-p) = (v\chi_{N\cap K} \vee 0)(-p)\} .$$

From Lemma 1.3, $\mu(D(e)) = 1$ for all maximal $\mu \in P(K)$. Letting $D = \bigcap_{n,j} D(e(F_{n,j}))$, D is Borel and $\mu(D) = 1$ for maximal $\mu \in P(K)$.

We will show that if $p \in D \cap Z\setminus\{0\}$ then (7.18) is valid. Letting $N_{n,j} = e(F_{n,j})W$, we have that $p \in N_{n,j}$ if and only if $N(p) \subseteq N_{n,j}$, or equivalently,

$$N(p) \cap E(K) \subseteq N_{n,j} \cap E(K) = F_{n,j} .$$

From the definition of $\bar{\varphi}$ we have that $\varphi(q) = \bar{\varphi}(p)$ for all $q \in N(p) \cap E(K)$, hence fixing $q \in N(p) \cap E(K)$, $p \in N_{n,j}$ if and only if $\varphi(q) \geq \frac{i}{2^n}$. It follows that

$$\begin{aligned} v(S_n p) &= \frac{1}{2^n} \Sigma_j\, v(e(F_{n,j})p) \\ &= \frac{1}{2^n} \Sigma_j\, \chi_{N_{n,j}}(p)v(p) \\ &= \frac{1}{2^n} \Sigma_j\, \chi_{F_{n,j}}(q)v(p) \\ &= f_n(q)v(p) . \end{aligned}$$

Thus for each $p \in D \cap Z\setminus\{0\}$,

$$v(Sp) = \varphi(q)v(p) = \bar{\varphi}(p)v(q) .$$

Suppose that $p_o \in Z\setminus\{0\}$. We claim that there is a maximal $\mu \in P(K)$ with $r(\mu) = p_o$ and $\operatorname{supp}\mu \subseteq N(p_o)$. Let $\nu \in P(K)$ be maximal with $r(\nu) = p_o$. From Lemma 4.4, $\nu_1 = \nu|_{N(p_o)}$ is a maximal measure on K with $r(\nu_1) = p_o$ and $\|\nu_1\| \leq 1$. From the Krein-Milman Theorem we may select a point $r \in E(K \cap N(p_o))$, and from (3.13), $r \in E(K)$. Then

$$\mu = \nu_1 + \tfrac{1}{2}(1 - \|\nu_1\|)(\delta(r) + \delta(-r))$$

is a probability measure with the desired properties. (We added the extra term since the barycentric calculus has been defined in terms of probability measures.)

Since μ is maximal, $\mu(Z) = 1$ (see [Ph, p.30]), and $\mu(\{0\}) = 0$ since otherwise μ will have obvious delations. In addition, $\mu(D) = \mu(N(p_o)) = 1$, hence letting

$$D_1 = D \cap N(p_o) \cap [Z \setminus \{0\}]$$

we have $\mu(D_1) = 1$. If $p \in D_1$, then $N(p) \subseteq N(p_o)$, hence if $q \in N_p \cap E(K)$, then $q \in N(p_o) \cap E(K)$, and

$$\bar{\varphi}(p) = \varphi(q) = \bar{\varphi}(p_o) .$$

Since $v \circ S \in \mathcal{A}(K)$ we have

$$\begin{aligned} v(Sp_o) &= \int v(s\,p)d\mu(p) \\ &= \int_{D_1} \bar{\varphi}(p)v(p)d\mu(p) \\ &= \bar{\varphi}(p_o)v(p_o) . \end{aligned}$$

Since this formula is obviously true if $p_o = 0 \in Z$, we have established (7.18), and we are done.

<u>Remark 7.10</u>: Let J be the map $J(p) = J_p$ of $E(K)$ onto Prim V. Since the structure topology on $E(K)$ is defined to be the pull-back of the structure topology on Prim V, a real function f on $E(K)$ will be structurally continuous if and only if f is of the form $g \circ J$, with g structurally continuous on Prim V. Thus Theorem 7.9 provides us with the one-to-one correspondence described in Theorem B of the Introduction.

8. L-structure in ordered Banach spaces

In the next section we will se how the M-ideals, the structure space, and the centralizer naturally arise in a large class of ordered Banach spaces (including the self-adjoint parts of C*-algebras) and in the (non-ordered) Lindenstrauss spaces. A preliminary task, however, will be to identify the "dual" notions" of L-ideal and Cunningham algebra. We begin with a discussion of the ordered Banach spaces which will appear in § 9 as the dual ordered Banach spaces. The ordered Banach spaces in these two sections have been investigated by a number of mathematicians (see [A,C-P,E,P,P,W]).

In this section we will assume that W is a real Banach space with closed unit ball K. Letting C be a convex cone, we will need the following definitions (the first was given in § 2):

A : C is <u>additive</u> if for all $p, q \in C$,
$$\|p+q\| = \|p\| + \|q\| .$$

G_1 : C is <u>1-generating</u> if
$$K = \mathrm{co}([K \cap C] \cup [K \cap -C]) .$$

It is quickly verified that C satisfies G_1 if and only if for each $p \in V$, there exist $q, r \in C$ for which

$$p = q - r , \quad \|p\| = \|q\| + \|r\| . \tag{8.1}$$

If the elements q and r are unique, we say that C is <u>uniquely</u> <u>1-generating</u>, and we write $q = p^+$ and $r = p^-$. It must be remenbered that p^+ is generally not the least upper bound of p and 0.

We say that an ordered Banach space W with cone W^+ is a

(<u>uniquely generated</u>) <u>A-space</u> if W^+ is closed, additive and (uniquely) 1-generated.

For purposes of illustration we list some examples of A-spaces. The first two are uniquely generated (see [D_2,§ 12.3.4]). This need not be the case for A_3 (e.g. let K be a square, or any polygone with two parallel edges).

A_1 A Kakutani L-space.

A_2 The ordered Banach space $(B_*)_{SA}$ of self-adjoint functionals in the pre-dual B_* of a von Neumann algebra B (we recall that B_* consists of the ultra-weakly continuous linear functions on B).

A_3 The ordered Banach dual of $A(K)$, the continuous affine functions on a compact convex subset K of a locally convex Hausdorff vector space.

In connection with A_2 it should be noted that the dual of a C*-algebra A coincides with the pre-dual of the von Neruann algebra $B = A^{**}$ (see [D_2,§ 12]). Turning to A_3 , the ordered Banach spaces $A(K)$ are just the Archimedean order unit spaces (see [A_2,Ch II,§ 1])

In the remainder of this section we will assume that W is an A-space. We examine first the cone W^+ .

<u>Lemma 8.1</u>. $(W^+)' = -W^+$

<u>Proof</u>. Since W^+ is additive, there is a facial cone C with $W^+ \subseteq C$ (Lemma 2.7 (b)). $-C$ is also a facial cone, and $(-C) \cap W^+ \subseteq (-C) \cap C = \{0\}$ (facial cones are proper), hence $-W^+ \subseteq -C \subseteq (W^+)'$. On the other hand, if $p \in (W^+)'$, choose $q,r \in W$ as in (8.1). Then since $(W^+)'$ is hereditary (see § 2),

and $q \precsim p$, we have that $q \in (W^+)' \cap W = \{0\}$, and $p = -r \in -W^+$.

Corollary 8.2. W^+ is a facial cone.

Proof. W^+ is additive, and since $W^+ = (-W^+)'$ it is hereditary. The assertion follows from Lemma 2.7 (c).

The above results enable us to characterize domination in W .

Proposition 8.3. Suppose that W is an A-space. Then

(a) If $q \in W^+$ and $p \in W$, then $p \precsim q$ if and only if $0 \leq p \leq q$.

If W is uniquely generated, we have

(b) If $p,q \in W$, then $p \precsim q$ if and only if $p^+ \leq q^+$ and $p^- \leq q^-$.

Proof. (a) From Corollary 8.2, W^+ is hereditary, hence $p \precsim q \in W^+$ implies that $p \in W^+$. On the other hand, since W^+ is additive, we have from the equivalence of (c) and (a) in Corollary 2.4 that $0 \leq p \leq q$ implies that $p \precsim q$.

(b) Suppose that $p \precsim q$. We have $p^+ \precsim p \precsim q$. From Theorem 2.9, there exist elements $q_1 \in W^+$ and $q_2 \in (W^+)' = -W^+$ with $q = q_1 + q_2$, $q_1 \mid q_2$, and $p^+ \precsim q_1$. Since W is uniquely generated $q_1 = q^+$, i.e. from (a), $p^+ \leq q^+$. Applying this to the relation $-p \precsim -q$,

$$p^- = (-p)^+ \leq (-q)^+ = q^- .$$

Conversely if $p^+ \leq q^+$ and $p^- \leq q^-$, then from (a), $p^+ \precsim q^+$, $p^- \precsim q^-$ and $-p^p \precsim -q^-$, hence since $q^+ \mid -q^-$ we have from (2.5) that $p \precsim q$.

We turn next to the hereditary subspaces of W. If N is a subset of W, we shall write N^+ for $N \cap W^+$. A subspace N of W is said to be an <u>order ideal</u> if $0 \leq q \leq p$ and $p \in N$ imply $q \in N$. It is readily verified that an equivalent condition is that N^+ be a face of the cone W^+.

<u>Proposition 8.4</u>. Suppose that W is an A-space and that N is a closed subspace of W, then the following are equivalent

(a) N is hereditary, i.e., $p \precsim q \in N$ implies $p \in N$

(b) N is an order ideal, and if $q \in N$ and $q = q_1 - q_2$ with $q_i \in W^+$ and $\|q\| = \|q_1\| + \|q_2\|$, then $q_1 \in N$.

If W is uniquely generated, we may replace (b) by

(b') N is an order ideal, and $q \in N$ implies $q^+ \in N$.

<u>Proof</u>. (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (a). Suppose that $p \precsim q \in N$. We may assume that $p \in W^+$, since in general $p = p_1 - p_2$ with $p_i \in W^+$, $p_i \precsim p$. From Theorem 2.9 we may select $q_i \in W^+$ with $q = q_1 - q_2$, $\|q\| = \|q_1\| + \|q_2\|$ and $p_1 \precsim q_1$, i.e., $0 \leq p_1 \leq q_1$. It follows from (b) that $p_1 \in N$. From the relation $-p \precsim -q$ we similarly conclude that $p_2 \in N$. (b') is an immediate consequence of (b).

For the pre-duals of von Neumann algebras there is a algebraic interpretation for the hereditary subspaces. Using Proposition 8.4 and analytic techniques, one can prove

<u>Proposition 8.5</u>. Suppose that B is a von Neumann algebra, and that $W = (B_*)_{SA}$ (see above). There is a natural one-to-one correspondence between

(a) The hereditary closed subspaces of W

(b) The projections in B

(c) The weak*-closed left ideals in B .

Proof. See [E_2], [Pr].

Before discussing the Cunningham algebra for W , we must introduce some order-theoretic notions.

If E is an ordered vector space, we order the linear operators on E by letting $S \leq T$ if $Sp \leq Tp$ for all $p \in E^+$. Following Buck [Buc] we define a linear operator T on E to be order-bounded if for some $\alpha \geq 0$, $-\alpha I \leq T \leq \alpha I$. We define $\mathcal{O}(E)$ to be the algebra of such operators, and we provide it with the ordering defined by the convex cone $\mathcal{O}^+(E) = \{T: 0 \leq T \leq \alpha I$ for some $\alpha \geq 0\}$. Although we shall not use this fact we note that that if E is Archimedean, i.e. $x \in E$, $y \in E^+$ and $x \leq \alpha y$ for all $a \geq 0$ imply $x \leq 0$, it follows that $\mathcal{O}(E)$ is an Archimedean order unit space and a commutative normed algebra (the latter essentially follows from Lemma 3.7) (see [Buck], [W], [AA_2]).

A subspace H of E is said to be an order summand of E if it is the range of a projection e on E for which $0 \leq e \leq I$ (see [C - P] and [W_1] for a discussion of this concept). This terminology is motivated by the fact that if e is such a projection, E is order-isomorphic to the vector sum $eE \oplus (I - e)E$ provided with the cone $eE^+ \oplus (I - e)E^+$.

Perdrizet has pointed out that there is a close connection between the notions of order summand and split face in this abstract setting [Pe_2]. Suppose that Q is a convex subset of a vector space. We say that a face F in Q is split if the complementary set F^c is convex (and thus a face), and if each $p \in Q$ has a unique representation $p = \alpha q + (1-\alpha)r$ with $q \in F$, $r \in F^c$ and $0 \leq \alpha \leq 1$.

Suppose that Q is a base for E^+, i.e., there is a linear function f on E which is strictly positive on $E^+ \setminus \{0\}$, such that $Q = \{p \in E^+ : f(p) = 1\}$. Then the map $H \mapsto H \cap Q$ defines a one-to-one correspondence between order summands in E and split faced in Q (see [Pe$_2$]).

Proposition 8.6. Suppose that W is an A-space, S is a linear operator on W, and $\alpha \geq 0$. Then $S \dashv \alpha I$ if and only if $0 \leq S \leq \alpha I$.

Proof. Suppose that $S \dashv \alpha I$. Then if $p \in W^+$, $Sp \dashv \alpha p$ and from Proposition 8.3 (a), $Sp \in W^+$. But we also have that $\alpha p - Sp \dashv \alpha p$, hence $\alpha p - Sp \in W^+$ and $0 \leq Sp \leq \alpha p$. Conversely suppose that $0 \leq S \leq \alpha I$. Then if $p \geq 0$, $0 \leq Sp \leq \alpha p$, i.e., from Proposition 8.3 (a), $Sp \dashv \alpha p$. For general $p \in W$, select q and r as in (8.1). Then from

$$\|Sq\| + \|\alpha q - Sq\| = \alpha\|q\|$$

$$\|Sr\| + \|\alpha r - Sr\| = \alpha\|r\|$$

we conclude

$$\|Sp\| + \|\alpha p - Sp\| \leq \alpha(\|q\| + \|r\|) = \alpha\|p\| ,$$

i.e., $S \dashv \alpha I$.

Corollary 8.7. The Cunningham algebra $\mathcal{C}(W)$ of an A-space W coincides as an ordered Banach algebra with the algebra of order-bounded operators $\mathcal{O}(W)$.

Corollary 8.8. The L-ideals of an A-space W coincide with the order summands of the underlying ordered vector space.

<u>Proof</u>. From Proposition 8.6, the projections e on W satisfying $e \dashv I$ coincide with those for which $0 \leq e \leq I$.

Suppose that W is an A-space. The map $p \longmapsto \|p\|$ is additive and positively homogeneous on W^+. It thus has an extension to a linear function f on W, and the positive face $S^+ = \{p \in W^+ : \|p\| = 1\}$ is a base for W^+ (W is a "base normed" space in the ssnse of Ellis [El_3]).

<u>Corollary 8.9</u>. If W is an A-space, then $N \mapsto N \cap S^+$ defines a one-to-one correspondence between the L-ideals in W and the split faces of S^+.

We will return to $\mathcal{G}(W)$ in § 9. We conclude this section with a result for L-spaces.

<u>Proposition 8.10</u>. Suppose that W is an L-space and N is a closed linear subspace of W. Then the following are equivalent

(a) N is hereditary

(b) N is an order ideal, and $N = N^+ - N^+$

(c) N is an L-ideal

(d) N is an order summand.

<u>Proof</u>. Due to Proposition 8.4 and Corollary 8.8, it suffices to prove that (b) $\Longrightarrow$ (d)

Let us assume (b). We claim that N must be a sublattice of W. It suffices to shaw that if $p \in N$, then $p^+ \in N$. Letting $p = q - r$, $r \in N^+$, we have that $p^+ \leq q$ (for L-spaces, p^+ <u>is</u> the supremum of 0 and p). Thus since N is an order ideal, $p \in N^+$.

We have that W is boundedly complete as a vector lattice (see [D_2,p.107]). It follows from [B_1, § II.1.5] that N is an order summand of W.

9. Applications of M-structure.

In this section we assume that V is a real Banach space, and that W is the dual of V. We let D (resp. B) be the closed (resp. open) unit balls in V, and K be the closed unit ball in W. We begin with a discussion of convex cones.

Let C be a convex cone in V and let $\leq$ be the ordering defined by C. We need the following definitions :

N: C is <u>normal</u> if

$$D = (C+D)\cap(C-D)$$

D: C is <u>directed</u> if for all $u,v\in D$ there exists a $w\in D$ with $u,v \leq w$.

D': C is <u>approximately directed</u> if for all $u,v\in B$ there exists a $w\in B$ with $u,v \leq w$.

We say that an ordered Banach space V with cone V^+ is an <u>F-space</u> if V^+ is closed, normal, and approximately directed. Examples of F-spaces include:

F_1: The ordered Banach space $C(X)$ of real continous functions on a compact Hausdorff space X.

F_2: A Kakutani M-space.

F_3: A simplex space (see $[E_3]$)

F_4: The ordered Banach space A_{SA} of self-adjoint elements in a C^*-algebra A.

F_5: The ordered Banach space $A(K)$ of real continuous affine functions on a compact convex subset K of a locally convex Hausdorff space.

If V is an F-space, we say that an element $I \in V$ is an <u>order unit</u> if $\|I\| \leq 1$ and $u \leq I$ for all $u \in D$. The simple observation that the archimedean order unit spaces (see § 3) are just the F-spaces with order unit, is stated more precisely in

<u>Lemma 9.1:</u> If V is an F-space with order unit I, then I is an archimedean order unit for the ordered vector space V and for all $v \in V$, $\|v\| = \|v\|_I$. Conversely, if V is an ordered vector space with archimedean order unit I and V is complete in the norm $\| \ \|_I$, then the corresponding ordered Banach space is an F-space with order unit I.

<u>Proof:</u> The verifications are all routine.

The following result is due to a number of mathematicians. An historical summary and elegant proofs may be found in [As_1]. If V is an ordered Banach space with cone V^+, we order the dual W by the cone $W^+ = \{p \in W : p|_{V^+} \geq 0\}$. If T is a subset of V (resp. W), we write $T^+ = T \cap V^+$ (resp. $T^+ = T \cap W^+$).

<u>Theorem 9.2:</u> Suppose that V is an ordered Banach space with closed cone V^+. Then V is an F-space if and only if V^* is an A-space.

<u>Corollary 9.3:</u> If V is an F-space, then

$$E(K) = E(K)^+ \cup [-E(K)^+] \tag{9.1}$$

<u>Proof:</u> Since W is 1-generating,

$$K = \mathrm{co}(K^+ \cup -K^+) \tag{9.2}$$

hence

$$E(K) \subseteq E(K^+) \cup E(-K^+) = E(K^+) \cup [-E(K^+)]$$

On the other hand, if $p \in E(K^+) \setminus \{0\}$, $\|p\| = 1$ since otherwise p would be a proper convex combination of 0 and $p/\|p\|$. Since W^+ is hereditary (Corollary 8.2),

$$\mathrm{face}_K\{p\} \subseteq C(p) \subseteq W^+,$$

hence

$$\mathrm{face}_K\{p\} = \mathrm{face}_{K^+}\{p\} = \{p\}$$

and $p \in E(K^+)$.

Regarding elements of V as functions on K, we have

<u>Corollary 9.4:</u> If V is an F-space, the map $v \mapsto v|_{K^+}$ is an isometric order isomorphism of V onto $A_0(K^+)$, the continuous affine functions on K^+ vanishing at 0.

<u>Proof:</u> The fact that $v \mapsto v|_{K^+}$ is an isometry is evident from (9.2). To show that it is onto, it suffices to show that any function $a \in A_0(K^+)$ extends to a weak * continuous linear function on W. This is proved in [K_2, Lemma 4.3] or [Pe_1].

<u>Remark:</u> By Corollary 9.4 the F-spaces are just the spaces that arise as spaces of real continuous affine functions vanishing at the top-point 0 of a universal cap of a cone (see [A_{s_2}]).

Although there does not seem to be a simple order theoretic characterization of M-domination in F-spaces, we do have an analogue of Proposition 8.3 (a):

Proposition 9.5 Suppose that V is an F-space. If $v \in V^+$ and $u \in V$, then $u \prec_M v$ if and only if $0 \leq u \leq v$.

Proof: From (9.2) and Corollary 9.4, it is evident that $v \mapsto v|_{E(K)^+}$ is an order isomorphism, hence $0 \leq u \leq v$ if and only if $0 \leq u(p) \leq v(p)$, i.e., $u(p) \prec v(p)$ for all $p \in E(K)^+$. From (9.1), the latter will occur if and only if $u \prec_M v$.

M-domination is adequate for describing the algebra $\mathcal{O}(V)$ of order bounded operators (see § 3) on an F-space V. In fact we have

Proposition 9.6: Suppose that V is an F-space, S is a linear operator on V and $\alpha \geq 0$. Then $S \prec_M \alpha I$ if and only if $0 \leq S \leq \alpha I$.

Proof: From Lemma 7.4 and Proposition 8.6 we have that $S \prec_M \alpha I$ if and only if $0 \leq S^* \leq \alpha I$, or equivalently

$$0 \leq (S^*p)(v) \leq \alpha p(v)$$

for all $p \in W^+$ and $v \in V^+$. The latter inequality may be rewritten

$$0 \leq p(Sv) \leq p(\alpha v).$$

Since V^+ is closed and $p \in W^+$ is arbitrary, this is the same as $0 \leq Sv \leq \alpha v$, i.e., $0 \leq S \leq \alpha I$.

Corollary 9.7: The centralizer $\mathcal{Z}(V)$ of an F-space coincides as an ordered Banach algebra with the algebra $\mathcal{O}(V)$ of order bounded operators.

<u>Corollary 9.8:</u> The M-summands of an F-space V coincide with the order summands of the underlying ordered vector space.

The order-theoretic characterization of M-ideals was considered by Perdrizet [Pe_2] and in the order-unit case by Alfsen and Andersen [AA_1]. Before describing some of their results (without proofs) it is necessary to relate the terminology used in those papers to ours.

Suppose that V is an F-space, and J is a closed subspace of V. Then from Corollary 8.8, J is an M-ideal if and only if J^o is an order-summand, i.e., J is a "hypostrict" ideal in the sense of Perdrizet. On the other hand, from Corollary 8.9, J is an M-ideal in V if and only if $J^o \cap S^+$ is a split face in S^+. Assuming that V has an order unit, S^+ is compact. A simple argument shows that the sets $J^o \cap S^+$ are the weak * closed split faces in S^+ (see [Pe_2, p.58]), i.e., the M-ideals are just the annihilators of weak * closed split faces in S^+. This is the context in which M-ideals are discussed in [AA_1].

If J is a subspace of an ordered vector space V, we give V/J the (possibly degenerate) ordering defined by the (possibly improper) cone $\pi(V^+)$, where π is the quotient map of V onto V/J. Given $v,w \in V$, we use the notation

$$[u,v] = \{w \in V: \ u \leq w \leq v\}.$$

In the following, (a) <=> (b) is due to Alfsen and Andersen [AA_1], and (a) <=> (c) to Perdrizet [Pe_2]. (Subspaces satisfying (i) - (iii) of (b) were first studied by Størmer, who termed then "Archimedean ideals" [S]).

Theorem 9.10: Suppose that V is an F-space with order unit I, and that J is a closed subspace of V. Then the following are equivalent:

(a) J is an M-ideal

(b) J satisfies each of the following conditions:

(i) J is positively generated.

(ii) J is an order ideal, i.e., given $j \in J$ and $v \in V$ with $0 \leq v \leq j$, it follows that $v \in J$ (equivalently, $\pi(V^+)$ is a proper cone).

(c) (iii) $\pi(I)$ is an Archimedean unit for V/J.

(iv) Given $v, w \in V^+$ and $\varepsilon > 0$,

$\pi([0,v]) \cap \pi([0,w]) \subseteq \pi([0,v+\varepsilon] \cap [0,w+\varepsilon])$.

(c) J satisfies each of the following conditions:

(I_1') If $j,k \in J$ and $v \in V$ are such that $j,k \leq v$, then for each $\varepsilon > 0$, there exists an $h \in J$ with $j,k \leq h \leq v + \varepsilon I$.

(I_2') If $u,s \in V^+$ and $h \in J$ are such that $h \leq u + v$, then for each $\varepsilon > 0$, there exist $j,k \in J$ with $h = j + k$ and $j \leq u, k \leq v + \varepsilon I$.

Let us now suppose that the Banach space V is itself the dual of a Banach space U.

Lemma 9.11: The adjoint map $T \mapsto T^*$ is an isometric isomorphism of $\mathcal{C}(U)$ into $\mathcal{Z}(V)$.

Proof: If e is an L-projection on U we have from Proposition 5.16 that e^* is an M-projection on V, hence $e \in \mathcal{Z}(V)$. Since

$\mathscr{C}(U)$ is the Banach algebra generated by the L-projections on U and $T \mapsto T^*$ is an anti-isomorphic isometry, the assertion follows.

The following result generalizes a theorem of Grothendieck (see Corollary 9.14). Our argument closely parallels the one that he gave [G, pp.555-556].

<u>Theorem 9.12:</u> Suppose V is an F-space with respect to some convex cone V^+ in V, and that V is the dual of a Banach space U. Then the map $T \mapsto T^*$ maps $\mathscr{C}(U)$ onto $\mathcal{Z}(V)$.

<u>Proof:</u> We first prove that V^+ is weak * closed. It suffices to show that $D^+ = V^+ \cap D$ is weak * closed [D.S., p.429]. Since norm-closed balls in V are weak * closed, it will suffice to prove that

$$D^+ = \bigcap \{D(v,1) : v \in D^+\},$$

where $D(v,r)$ is the norm-closed ball with center v and radius r. Given u,v D^+ we have $-v \le u-v \le u$, hence from the normality of W^+, $\|u-v\| \le 1$, and D^+ is contained in the intersection on the right. Conversely, suppose that u lies in the intersection. Then $u \in D = D(0,1)$, and we must show that $u \in V^+$. If $u \notin V^+$, then from Corollary 9.4, there is an element $p \in K^+$ with $u(p) < 0$. Letting $\varepsilon = -u(p)$ we may select an element t in the open unit ball B with $t(p) > 1 - \varepsilon$. Since V is approximately directed, we may select $t_1 \in B$ with $t_1 \ge t, 0$. We have

$$(t_1 - u)(p) > 1 - \varepsilon + \varepsilon = 1,$$

hence $\|t_1 - u\| > 1$, contradicting the fact that u is in the intersection.

We prove next that the order on V^+ is boundedly complete. Suppose that $\{v_\gamma\}_{\gamma\in\Gamma}$ is an increasing net in V^+ and that $\|v_\gamma\| \leq 1$.

Since V^+ is normal, $v_\gamma \in D$, and we may extract a subnet $\{v_\delta\}_{\delta\in\Delta}$ converging weakly* to an element $v \in D$. It suffices to show that v is a least upper bound for the set $\{v_\gamma\}$. Fixing $\gamma\in\Gamma$, we may select $\delta_0 \geq \gamma$. Then $v_\delta \geq v_\gamma$ for $\delta \geq \delta_0$ and since $v_\gamma + W^+$ is weak* closed, $v \geq v_\gamma$. Given $w \geq v_\gamma$ for all $\gamma\in\Gamma$, $w - W^+$ is weak* closed and $w \geq v$.

Since B^+ is directed in its ordering, we may let I be the least upper bound for B^+. It is quickly verified that I is an order unit for V. On the other hand, since $\mathcal{Z}(V)$ may be identified with the order-bounded operators on V (Corollary 9.7), one sees that $\mathcal{Z}(V)$ is also boundedly complete. In particular, it is the norm closed linear span of its projections. Thus to show that $T \mapsto T^*$ maps $\mathcal{C}(U)$ onto $\mathcal{Z}(V)$, it suffices to prove that each projection in $\mathcal{Z}(V)$, i.e., each M-projection on V, is in the range of this map.

Suppose that e is an M-projection on V. It will suffice to show that e is weak* continuous, since then there will be a projection f on U with $e = f^*$, and from Prop.5.15, $f \in \mathcal{C}(U)$. In turn, it suffices to prove that eV and $(I-e)V$ are weak* closed. (see [Pe_2, p.80]). Since e and $I-e$ have the same properties, we consider only eV.

From ([Pe_2, p.80] or [G, p.556]) it suffices to show that $eV \cap D$ is weak* closed. Suppose that $\{p_\gamma\}$ is a net in $eV \cap D$, converging weak* to an element $q \in V$. Using the order identity in V (see above),

$$-I \leq p_\gamma \leq I$$

hence applying the projection e,

$$-e(I) \leq p_\gamma \leq e(I)$$

since $0 \leq e \leq I$. Since the sets $-e(I) + V^+$ and $e(I) - V^+$ are weak* closed, it follows that

$$-e(I) \leq p \leq e(I).$$

Applying the operator $I - e$, we conclude

$$0 \leq p - ep \leq 0,$$

hence $p = ep \in eW$.

Corollary 9.13: If V is a dual F-space, then the M-summands in V are weak* closed.

Corollary 9.14 (Grothendieck [G]): Let $C_0(X)$ be the continuous functions vanishing at ∞ on a locally compact space X with the supremum norm. If $C_0(X)$ is the dual of a Banach space, then multiplication is weak* continuous in each variable.

Turning to operator algebra, suppose that $\mathfrak{A}$ is a von Neumann algebra. $\mathfrak{A}$ has a unique pre-dual $\mathfrak{A}_*$, and we provide $\mathfrak{A}$ with the corresponding weak* topology. The real subspace $V = \mathfrak{A}_{SA}$ of selfadjoint elements in $\mathfrak{A}$ is an F-space with order unit, i.e., an Archimedean order unit space. If $v \in \mathfrak{A}$, we let $M(v)$ be the corresponding multiplication transformation of $\mathfrak{A}$, i.e., if $w \in \mathfrak{A}$, we define $M(v)w = vw$. If v is in the center of $\mathfrak{A}$ and v is self-adjoint, then $M(v)V \subseteq V$. We have

Lemma 9.15: The centralizer of V consists of the operators $M(v)|_V$ with v a self-adjoint element of the center of $\mathfrak{A}$.

Proof: Se [An_1], which includes a more general result, or [W], which has a particularly elegant proof.

Proposition 9.16: If V is the self-adjoint part of a von Neumann algebra $\mathfrak{A}$, then the M-summands of V are just the subspaces of the form $J \cap V$ with J a weak* closed two-sided ideal in $\mathfrak{A}$.

Proof: The weak* closed two-sided ideals in $\mathfrak{A}$ are those subspaces of the form $e\mathfrak{A}$, with e a central self-adjoint projection in $\mathfrak{A}$ (see [D_2]). From Lemma 9.15, the intersections $J \cap V = eV$ are the M-summands on V.

Now suppose that A is a C^*-algebra, and that $V = A_{SA}$ is the F-space of self-adjoint elements. We have that the second complex dual $\mathfrak{A} = A^{**}$ is a von Neumann algebra (see [D_1, § 12]), and we may identify the second real dual V^{**} with $\mathfrak{A}_{SA}$. If $T \in \mathcal{Z}(V)$, then $T^* \in \mathcal{C}(V^*)$ (by definition), and $T^{**} \in \mathcal{Z}(V^{**})$ (see Lemma 9.11). If V has a unit I, then $T^{**}(I)$ will lie in the center of $\mathfrak{A}$, and thus in the center of A. Conversely, regarding A as a subalgebra of $\mathfrak{A}$, any element of the center of A will be in the center of $\mathfrak{A}$. Using the multiplier notation, we conclude

Corollary 9.17: If V is the self-adjoint part of a C^*-algebra A with unit, then the centralizer of V consists of the operators

$M(v)|_V$ with v a self-adjoint element of the center of A.

Proposition 9.18: If V is the self-adjoint part of a C^*-algebra A, then the M-ideals in V are just the subspaces of the form $J \cap V$ with J a uniformly closed two-sided ideal in A.

Proof: If J_1 is an M-ideal in V, let $J_1{}^{o} \subseteq V^*$ and $J_1{}^{oo} \subseteq V^{**}$ be the first and second real annihilators of J_1. Letting e be the L-projection on V^* with $J_1{}^{o} = eV^*$, it follows that $J_1{}^{oo} = (I-e^*)V^{**}$ (see (5.37)), hence from Proposition 5.16, J^{oo} is an M-summand in V^{**}. If we let $\bar{J}$ be the weak* closed two-sided ideal in $\mathfrak{A}$ with $J_1{}^{oo} = \bar{J} \cap V^{**}$, we have that $J = \bar{J} \cap A$ is a closed two-sided ideal in A with $J_1 = J \cap V$.

Conversely if J is a closed two-sided ideal in A, the weak* closure $\bar{J} = J^{oo}$ in $\mathfrak{A}$ is a weak* closed two-sided ideal, and there is a central projection f in $\mathfrak{A}$ with $J = f\mathfrak{A}$. From Lemma 9.15, $f_1 = M(f)|_{V^{**}}$ is an M-projection. Since multiplication is weak* continuous in each variable in $\mathfrak{A}$, there is a projection e on V^* with $e^* = f_1$. From Proposition 5.16, e is an L-projection, and $(J \cap V)^{o} = (1-e)V^*$ is an L-ideal. We conclude that $J \cap V$ is an M-ideal in V.

Corollary 9.19: The primitive M-ideals in V are just the subspaces of the form $J \cap V$, with J a primitive ideal in A.

Proof: If $p \neq 0$ is an extreme element of K^+ (i.e., p is a "pure state"), let L^p be the corresponding irreducible representation of A with cyclic vector X_p. From Corollary 9.3 we have

$$E(K) = E(K^+) \cup [-E(K^+)] \quad \{0\}.$$

Since $J_p = J_{-p}$, it suffices to prove that if $p \in E(K^+)$, then

$$J_p = \ker L^p \cap V.$$

Since $H_p = \ker L^p \cap V$ is an M-ideal in V with $H_p \subseteq \ker p$, we have $H_p \subseteq J_p$. Conversely, let I_p be a closed two-sided ideal in A with $J_p = I_p \cap A$. Since $I_p^* = I_p$, it is evident that $I_p = J_p + iJ_p$. Regarding p as an element of A^*, it is complex linear, and $p|_{I_p} = 0$. For any $v \in I_p$ and $w \in A$, we have

$$\|L^p(v)L^p(w)\chi_p\|^2 = p(w^*v^*vw) = 0$$

since I_p is a two-sided ideal. Since χ_p is cyclic, $L^p(v) = 0$, and $v \in \ker L^p$. Thus $I_p \subseteq \ker L^p$, and $J_p \subseteq H_p$.

Finally, suppose that V is a Lindenstrauss space, i.e., that it is a (non-ordered) Banach space for which the dual W is (isometric to) a Kakutani L-space. The structure of such Banach spaces was investigated in $[E_1]$, and we wish to indicate how that theory fits into our present development.

Suppose that N is an L-ideal in W. Then the set $H = N \cap K$ has the following properties:

B_1 H is convex and symmetric

B_2 If $p \neq 0$ is in H, then so is $p/\|p\|$.

B_3 If $q \in H$ and $p \prec q$, then $p \in H$.

Thus H is a <u>biface</u> in the terminology of $[E_1]$. Conversely if H is a uniformly closed biface, then its linear span N is hereditary, hence since W is an L-space, it is an L-ideal.

In $[E_1]$, a subset F of $E(K)$ was defined to be "structurally closed" if it was of the form $E(H)$, $H \neq \{0\}$ a weak* closed biface, or if $F = \emptyset$. Letting $H = N \cap K$, N the corresponding weak* closed L-ideal, $E(H) = N \cap E(K)$, and we conclude that we have the topology introduced on $E(K)$ in § 6.

10. Some historical notes and open problems.

The identification of a Banach space V with the space $A_0(K)$ of weak* continuous linear functions on the closed unit ball K of the dual space $W = V^*$ goes back to the early days of Banach space theory. (The non-trivial part of it is Banach's theorem that for linear functionals on W, bounded weak* continuity implies weak* continuity). After the development of the representation theory of Choquet and Bishop de Leeuw it became possible to study V by analysis on the symmetric compact convex set K. Although formally a special case of convexity theory, the investigation of symmetric convex sets is really a field of its own, since the relevant problems are different. Instead of studying the space $A(K)$ of all continuous affine functions, one studies the subspace $A_0(K)$ of those that vanish at 0 (the center of symmetry). In fact the "symmetric" theory may also be regarded as a generalization, since every compact convex set K_0 can be embedded in an affine hyperplane in such a way that $A(K_0) \cong A_0(K)$ where $K = co(K_0 \cup -K_0)$. In contrast with $A(K)$, there is no natural partial ordering compatible with the linear structure of $A_0(K)$, and this makes it necessary to apply rather different methods for the investigation of $A_0(K)$ than for $A(K)$.

A systematic use of $A_0(K)$, together with geometric and analytic properties of K, was used by Lazar and Lindenstrauss [L-L], Lazar [La_1] and Lindenstraus and Wulbert [L-W], for the study of Banach spaces whose duals are L^1-spaces (Lindenstrauss spaces). The importance of such spaces was first recognized by Grothendieck [G], and recent investigations show that they form a natural non-ordered generalization of the ordered spaces A(K) with K a Choquet simplex, as well as the somewhat more general simplex spaces (possibly without unit) which were introduced by Effros $[E_3]$. The domination ordering $\prec$ used in section 2 was introduced in $[E_1]$ for the study of the non-ordered Lindenstrauss spaces.

The structure of compact convex sets has recently been studied by several authors, e.g., by Asimow $[As_1]$, Ellis $[El_1]$, T. B. Andersen and E. Alfsen $[AA_1]$,$[A_3]$ and Ng $[N_1]$. The "splitting" property of closed faces of a simplex K has been known for quite a long time, and the "facial topology" or structure topology on E(K) was introduced by Effros in analogy with the corresponding topology in C^*-algebra. The notion of a split face was developed independently by Perdrizet $[Pe_2]$ and T.B.Andersen and E. Alfsen $[AA_1]$, and it made it possible to transfer the notion of a facial topology to general compact convex sets.

The fundamental Lemma 4.1 of the present paper is the "symmetric" analogue of the characterization of split faces by affinity of envelopes (see $[AA_1]$). However, the lack of ordering makes it impossible to apply the filtering argument used for split faces. A "balancing technique" somewhat analogous to that of the present proof was used by Lazar in $[La_1]$, (see also Lazar and Lindenstrauss [L-L]).

The characterization of weak* closed L-ideals by annihilating measures (Theorem 4.5) is almost identical with the similar characterization for split faces. $[A_2$,p.136]. The latter has proved useful in applications to complex function algebras [H], and it seems likely that Theorem 4.5 may be the most convenient criterion to determine weak* closed L-ideals in applications.

The "sharp" ($\leq$) dominated extension theorem given in Theorem 5.4 is one of the main results of the paper. It had precedents in the Edwards' separation theorem for simplexes [Ed], the Andersen extension theorem for split faces of a compact convex set [An_2] (a weaker form of this result was used in [AA_1]), and the Lazar-Lindenstrauss extension theorem for Lindenstrauss spaces [L-L].

The characterization of M-ideals by intersection properties with balls (Theorem 5.8-5.9) is new (as is the notion of an M-ideal). Nonetheless, they have the flavor of Lindenstrauss's intersection characterizations for Lindenstrauss spaces. In addition, the use of an extension theorem to prove an intersection property in (a) => (c) of Theorem 5.8 was inspired by a similar argument of Lazar in [La_1].

The notion of center (or "multiplier") was defined in the context of partially ordered Banach spaces by Wils [W], and for A(K)-spaces independently by E. Alfsen and T.B. Andersen [AA_1]. The connection with structurally continuous functions and order bounded operators was established in these papers. The latter generalizes similar theorems of Effros for simplex spaces and Lindenstrauss spaces [E_4], [E_1] and also the earlier Dauns-Hofmann theorem for C^*-algebras [D-H] (see also [D_3]), while the former (as well as the very notion of an order bounded operator) goes back to Buck [Buc]. Besides the results quoted above, Wils' paper also explains the connection between "central idempotents" and order summands, and he develops a continuous analogue of a direct ordered sum decomposition, which generalizes Sakai's representation of states on a C^*-algebra by central measures [Sa].

The fact that the closed two-sided ideals in a C^*-algebra A correspond to the Archimedean ideals in A_{SA} (see the discussion preceding Theorem 9.10) is due to Størmer [S]. Alfsen and Andersen [AA_1] pointed out that the latter ideals must also satisfy (b) - (iv) of Theorem 9.10. On the other hand Combes and Perdrizet proved that the closed two-sided ideals correspond in the same way to the subspaces of A_{SA} satisfying I_1' and I_2' of Theorem 9.10 (c) [C-P].

We conclude with some problems that we have left open.

1. Repeating a problem that arose in section 5, suppose that N is a closed subspace of a Banach space W such that N^o is an M-ideal in W^*, does it follow that N^o is an L-ideal in W^* ? Perdrizet proved that this is the case if W is an A-space [Pe_2].

2. If V is the dual of a Banach space U, must each weak* closed M-ideal in V be an M-summand ?

3. Suppose that J is an M-ideal in V, and that D_1 and D_2 are closed balls with $D_1 \cap D_2 \neq \emptyset$ (i.e., we do not assume interior intersection), and $D_1 \cap J \neq \emptyset$. Does it follow that $D_1 \cap D_2 \cap J \neq \emptyset$? We note that the counter-example in Remark 5.10 used three balls. Some evidence that this is the case exists in C^*-algebra theory. Pedersen proved in [P] that if J is an M-ideal in the self-adjoint part V of a C^*-algebra, then it satisfies condition I_2 of Combes and Perdrizet [C-P] (this is just I_2' with $\varepsilon = 0$ in the statement of Theorem 9.10). It is not difficult to show that I_2 would be the consequence of the "non-interior" 2-ball condition.

Perhaps one could prove this in the context of C*-algebras.

4. Lazar and Lindenstrauss proved that if V is a Lindenstrauss space, then $\check{g}(0) \geq 0$ is a necessary and sufficient condition for the extension in Theorem 5.4 [L-L]. Is this still the case for the larger class of Banach spaces satisfying the "3,2 intersection property" (this includes the L-spaces - see [Li])? One small piece of evidence for this is that one can prove directly that for such spaces the M-ideals satisfy the "non-interior" 2-ball property.

5. Is Theorem 9.12 valid for arbitrary real Banach spaces V ?

6. Grothendieck used Corollary 9.14 to prove that $C_o(X)$ can have at most one pre-dual Banach space U . Is the latter true for arbitrary F-spaces? Sakai proved that this is the case for the self-adjoint elements of a C*-algebra [Sa_2].

7. In Lemma 4.3 we proved that $C' \cap S$ is weak* Borel in K . Is this also the case for $C' \cap K$?

8. Theorem 7.9 can be simplified if V is separable. In this case one simply has to approximate the function $p \to \varphi(p)v(p)$ by step functions on $E(K)$. One then uses Lemma 4.1 to extend the corresponding characteristic functions of closed sets to L-projections on W . The key point is that since $E(K)$ carries maximal measures, one can use the barycentric calculus to show that the corresponding operators converge. By a known theorem of Bishop-de Leeuw (see, e.g.,[Ph, p.24]), this technique could be extended to the non-separable case if one knew that the approximating functions were Baire rather than just Borel.

9. A rather less specific problem is to find analogues of the results in this paper for complex Banach spaces. We have some results in this direction, which will appear in a subsequent paper.

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